M-IDEALS IN $H^\infty(\mathbb{D})$

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Abstract. This article intends to initiate an investigation into the structure of M-ideals in $H^\infty(\mathbb{D})$, where $H^\infty(\mathbb{D})$ denotes the Banach algebra of all bounded analytic functions on the open unit disc $\mathbb{D}$ in $\mathbb{C}$. We introduce the notion of analytic primes and prove that M-ideals in $H^\infty(\mathbb{D})$ are analytic primes. From Hilbert function space perspective, we additionally prove that M-ideals in $H^\infty(\mathbb{D})$ are dense in the Hardy space. We show that outer functions play a key role in representing singly generated closed ideals in $H^\infty(\mathbb{D})$ that are M-ideals. This is also relevant to M-ideals in $H^\infty(\mathbb{D})$ that are finitely generated closed ideals in $H^\infty(\mathbb{D})$. We analyze p-sets of $H^\infty(\mathbb{D})$ and their connection to the Šilov boundary of the maximal ideal space of $H^\infty(\mathbb{D})$. Some of our results apply to the polydisc. In addition to addressing questions regarding M-ideals, the results presented in this paper offer some new perspectives on bounded analytic functions.

Contents

1. Introduction

The objective of this work is to integrate two classic concepts that are rich and independent areas of study: M-ideals in Banach spaces and the Banach algebra $H^\infty(\mathbb{D})$. In 1972, Alfsen and Effros [3] introduced the concept of M-ideals (see Definition 1.1) as a means of extending the utility of closed two-sided ideals in C^*-algebras to Banach spaces. They proved, in particular, that a closed subspace of a real Banach space is an M-ideal if and only if the closed subspace satisfies the 3-ball property. Beginning with this geometric implication, the concept of M-ideals evolved into one of the most useful tools in Banach space theory (specifically in the geometry of Banach spaces) and eventually became a subject in its own right. See the monograph by Harmand, Werner, and Werner [18], and also see the series of papers [6, 10, 11, 14, 16, 17, 19, 35, 44, 45, 46] and the references therein.

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The second topic of this paper, $H^\infty(\mathbb{D})$, is even more classic (see Bers [5] and Kakutani [30], and the survey by Gamelin [12]). Recall that $H^\infty(\mathbb{D})$ stands for the space of bounded analytic functions on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, that is

$$H^\infty(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < \infty\}.$$

This is a commutative Banach algebra with respect to the pointwise product and under the supremum norm

$$\|f\| = \sup_{z\in\mathbb{D}} |f(z)| \qquad (f \in H^\infty(\mathbb{D})).$$

$H^\infty(\mathbb{D})$ is one of the most important Banach algebras that plays a key role in the theory of Hilbert function spaces and Banach spaces of analytic functions [26]. And because of its analytic nature, it fits well between commutative C^*-algebras and generic commutative Banach algebras. For instance, if $\mathcal{M}(H^\infty(\mathbb{D}))$ denote the maximal ideal space of $H^\infty(\mathbb{D})$, then the Gelfand map $\Gamma : H^\infty(\mathbb{D}) \to C(\mathcal{M}(H^\infty(\mathbb{D}))$ defined by

$$\Gamma(f) = \hat{f} \qquad (f \in H^\infty(\mathbb{D})),$$

is an isometry: a prototype feature of commutative C^*-algebras (see Section 2 for more details). However, despite being concrete and popular among commutative Banach algebras, this space raises fundamental problems for which there are no definitive solutions. The structure of the closed ideals in $H^\infty(\mathbb{D})$, for example, is still obscure (however, see [22], and also see [2, 9, 34]). On one hand, this space has undergone extensive research in the context of Banach spaces of analytic functions (cf. [8, 28, 31, 32, 33]). On the other hand, as far as our knowledge extends, $H^\infty(\mathbb{D})$ has not been investigated in any way from the standpoint of M-ideals. And in this paper, this is exactly what we do.

Note that $H^\infty(\mathbb{D})$ is also an example of a uniform algebra, and the classification of M-ideals in uniform algebras, which often involves maximal ideal spaces, is well known (see Theorem 2.3). Unfortunately, since the maximal ideal space and the Šilov boundary of $H^\infty(\mathbb{D})$ are hard to understand, uses of the general classification (which also includes peak sets and p-sets) on the uniform algebra $H^\infty(\mathbb{D})$ only give abstract results. In fact, as we proceed further, along with M-ideals we will also study the delicate structure of p-sets of the maximal ideal space of $H^\infty(\mathbb{D})$.

We now go over the definition of M-ideals. Hereafter, all the Banach spaces are over the field of complex numbers. Furthermore, projections on a Banach space refer to bounded linear operators P on that space such that

$$P^2 = P.$$

Definition 1.1. *A closed subspace C of a Banach space X is said to be an M-ideal in X if there exists a projection $P : X^* \to X^*$ such that*

$$ranP = C^\perp,$$

and

$$\|Px^*\| + \|(I - P)x^*\| = \|x^*\|,$$

for all $x^* \in X^*$.

The projection P above is commonly known as an *L-projection.* Also, we denote by $C^\perp$ the annihilator of C:

$$C^\perp = \{x^* \in X^* : x^*|_C \equiv 0\}.$$

We now proceed directly to elucidating the main contribution of this paper. The issue of classifying or representing M-ideals in $H^\infty(\mathbb{D})$ is highly intricate, and its complexity is comparable to the hierarchy of maximum ideal spaces in $H^\infty(\mathbb{D})$. Indeed, our first result refers to the "size" of M-ideals in $H^\infty(\mathbb{D})$ (see Proposition 3.3):

Proposition 1.2. *Each maximal ideal in $H^\infty(\mathbb{D})$ that corresponds to a complex homomorphism in the Šilov boundary of $H^\infty(\mathbb{D})$ is an M-ideal in $H^\infty(\mathbb{D})$.*

Denote by $\partial_S H^\infty(\mathbb{D})$ the Šilov boundary of the maximal ideal space $\mathcal{M}(H^\infty(\mathbb{D}))$ of $H^\infty(\mathbb{D})$. The above result says that for each $\varphi \in \partial_S H^\infty(\mathbb{D})$, the singleton set $\{\varphi\}$ is a p-set of $H^\infty(\mathbb{D})$, and hence the maximal ideal corresponding to each complex homomorphism in $\partial_S H^\infty(\mathbb{D})$ is an M-ideal in $H^\infty(\mathbb{D})$. Therefore, it seems overly generic to have a detailed description of every M-ideal in $H^\infty(\mathbb{D})$.

As already hinted, there is an inextricable link between the theory of M-ideals in a uniform algebra and the structure of p-sets within the same uniform algebra. This paper presents new insights on p-sets of $H^\infty(\mathbb{D})$ and their application to M-ideals in $H^\infty(\mathbb{D})$. For instance, Theorem 4.1 states:

Theorem 1.3. *Let Q be a nonempty subset of $\mathcal{M}(H^\infty(\mathbb{D}))$. If Q is a p-set of $H^\infty(\mathbb{D})$, then*

$$Q \cap \partial_S H^\infty(\mathbb{D}) \neq \emptyset.$$

It is noteworthy to mention that, in contrast to the maximal ideal spaces of $H^\infty(\mathbb{D})$, evaluation functions do not correspond to M-ideals in $H^\infty(\mathbb{D})$, and they play a less significant role (see Corollary 4.2).

Next, we introduce an analytic counterpart of the algebraic notion of prime ideals in commutative rings. Before getting into the specifics of this new idea, it is important to go over a key feature of bounded analytic functions. Given a function $f \in H^\infty(\mathbb{D})$, we denote by $\tilde{f}$ the radial limit extension of f:

$$\tilde{f}(z) = \lim_{r\to 1^-} f(rz) \qquad (z \in \mathbb{T}\ a.e.). \tag{1.1}$$

The above limit's existence is an intricate implementation of Fatou's theorem [26, page 34].

Definition 1.4. *A proper nontrivial closed subspace J of $H^\infty(\mathbb{D})$ is called an analytic prime whenever it meets the following condition (see Definition 5.1): Given f and g in $H^\infty(\mathbb{D})$, if*

$$fg \in J,$$

and

$$|\tilde{f}(z)| > \delta \qquad (z \in \mathbb{T}),$$

for some $\delta > 0$, then

$$g \in J.$$

It is now reasonable to look for examples of analytic primes in $H^\infty(\mathbb{D})$. In Theorem 5.2, we prove that M-ideals in $H^\infty(\mathbb{D})$ are analytic primes in $H^\infty(\mathbb{D})$:

Theorem 1.5. *Proper nontrivial M-ideals in $H^\infty(\mathbb{D})$ are analytic primes.*

We also connect M-ideals with inner functions. A function $f \in H^\infty(\mathbb{D})$ is called *inner* if

$$|\tilde{f}(z)| = 1,$$

for all $z \in \mathbb{T}$ a.e. [26, page 62]. It is clear that the space of inner functions forms a multiplicative set in $H^\infty(\mathbb{D})$. In Corollary 5.3, we prove that:

Theorem 1.6. *A proper and nontrivial M-ideal in $H^\infty(\mathbb{D})$ never intersects the space of inner functions.*

We also discuss some relationships between M-ideals in $H^\infty(\mathbb{D})$ and the theory of Hilbert function spaces as well as classical operator theory. The Hilbert function space here is the Hardy space

$$H^2(\mathbb{D}) := \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\| := \Big(\sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^2 d\mu(z) \Big)^{\frac{1}{2}} < \infty \Big\},$$

where $d\mu$ denotes the normalized Lebesgue measure on $\mathbb{T}$. In terms of radial limits, $H^2(\mathbb{D})$ is also the Hilbert space consisting of all analytic functions on $\mathbb{D}$ with square summable Taylor coefficients (see Section 5 for more details):

$$H^2(\mathbb{D}) = \{ f = \sum_{m=0}^{\infty} \alpha_m z^m \in \mathrm{Hol}(\mathbb{D}) : \|f\| = \Big(\sum_{m=0}^{\infty} |\alpha_m|^2 \Big)^{\frac{1}{2}} < \infty \}.$$

The integral representation of $H^2(\mathbb{D})$ implies that the inclusion map

$$i : H^\infty(\mathbb{D}) \hookrightarrow H^2(\mathbb{D}),$$

is a contraction. In Theorem 5.5, we prove a Hilbert space property of M-ideals in $H^\infty(\mathbb{D})$. More specifically:

Theorem 1.7. *If J is a nontrivial M-ideal in $H^\infty(\mathbb{D})$, then*

$$\overline{J}^{H^2(\mathbb{D})} = H^2(\mathbb{D}).$$

This norm density property is a notable characteristic of M-ideals in $H^\infty(\mathbb{D})$. This indeed reminds us of the norm density of the ring of polynomials:

$$\overline{\mathbb{C}[z]}^{H^2(\mathbb{D})} = H^2(\mathbb{D}).$$

Now we turn to the principal ideals or singly generated ideals in $H^\infty(\mathbb{D})$. Given $f \in H^\infty(\mathbb{D})$, we denote by $I_{H^\infty(\mathbb{D})}(f)$ the closed ideal of $H^\infty(\mathbb{D})$ generated by f. Therefore

$$I_{H^\infty(\mathbb{D})}(f) = \overline{\{ fg : g \in H^\infty(\mathbb{D}) \}}^{H^\infty(\mathbb{D})}. \tag{1.2}$$

We remind the reader that the classification of principle ideals in $H^\infty(\mathbb{D})$ is unknown. Our goal here is to study the M-ideal structure of $I_{H^\infty(\mathbb{D})}(f)$. Our first necessary condition for such a property is rather operator theoretic. In fact, as a consequence of Theorem 1.7, we have the following feature about Toeplitz operators on the Hardy space. Let $\varphi \in H^\infty(\mathbb{D})$. The *Toeplitz operator* with symbol φ is the bounded linear operator T_φ on $H^2(\mathbb{D})$ defined by

$$T_\varphi f = \varphi f \qquad (f \in H^2(\mathbb{D})).$$

If $I_{H^\infty(\mathbb{D}^n)}(\varphi)$ is an M-ideal in $H^\infty(\mathbb{D}^n)$, then (see Corollary 5.6)

$$\ker T_\varphi^* = \{0\}.$$

It is worth noting that $\ker T_\varphi^* = \ker T_{\bar{\varphi}}$. Kernels of Toeplitz operators are recognised as complex objects and a subject of independent interest [24, 42, 43]. Therefore, M-ideals in $H^\infty(\mathbb{D})$ touch on the delicacy of the kernels of Toeplitz operators.

We continue with more Hilbert function space techniques and results. We need to recall the inner-outer factorization of functions in $H^\infty(\mathbb{D})$; an essential result in functional analysis, particularly in the analysis of bounded analytic functions and Hilbert function space theory:

Let $f \in H^\infty(\mathbb{D})$ (or even $f \in H^2(\mathbb{D})$) be a nonzero function. Then there exist an inner function f_I and an outer function f_O such that [26, page 67]

$$f = f_I f_O, \tag{1.3}$$

on $\mathbb{D}$. Moreover, this factorization is unique up to a constant of modulus one. Recall that a function $f \in H^2(\mathbb{D})$ is *outer* if

$$\overline{\operatorname{span}\{z^m f : m \geq 0\}}^{H^2(\mathbb{D})} = H^2(\mathbb{D}).$$

We have already talked about the role of inner functions in M-ideals. The following is the second application of Theorem 1.7, which yields a significant consequence that links M-ideals with outer functions (see Corollary 5.8 for more details):

Theorem 1.8. *Let $f \in H^\infty(\mathbb{D})$. If $I_{H^\infty(\mathbb{D})}(f)$ is an M-ideal in $H^\infty(\mathbb{D})$, then f is an outer function.*

All the results from Theorem 1.5 to the theorem stated above are applicable in the polydisc setting and are consolidated in Section 5.

The above result also calls on the question of the extent to which the converse is true. The answer to this problem is apparently difficult. However, we introduce a large class of bounded analytic functions for which the converse is true. To clarify this class, it is necessary to introduce the notion of essential zeros of $H^\infty(\mathbb{D})$-functions, which may be a concept of independent significance.

Definition 1.9. *We say that $z \in \mathbb{T}$ is an essential zero of an outer function $f \in H^\infty(\mathbb{D})$ if, for any $\varepsilon > 0$ and every open neighborhood $N_z \subset \mathbb{T}$ of z, we have the following:*

$$\mu(|\tilde{f}|^{-1}(0, \varepsilon) \cap N_z) \neq 0.$$

We let $Z_{\mathbb{T}}(f)$ denote the essential zeros of the outer function f.

Clearly, if $Z_{\mathbb{T}}(f) = \emptyset$, then f is invertible. The example presented in (6.4) serves to illustrate the computation of zero sets of outer functions. The notion of zero sets of functions in $H^\infty(\mathbb{D})$ relies on the outer part as follows (see (1.3)): The *zero set* of a function $f \in H^\infty(\mathbb{D})$ is defined by

$$Z_{\mathbb{T}}(f) = \{z \in \mathbb{T} : z \in Z_{\mathbb{T}}(f_O)\}.$$

Definition 1.10. *Define $Z^\infty(\mathbb{D})$ as the set of all functions f in $H^\infty(\mathbb{D})$ that have a continuous extension to $Z_{\mathbb{T}}(f)$.*

It is also evident that $A(\mathbb{D}) \subseteq Z^\infty(\mathbb{D})$, where $A(\mathbb{D})$ is the disc algebra defined by

$$A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \in \operatorname{Hol}(\mathbb{D})\}. \tag{1.4}$$

On the other hand, $Z^\infty(\mathbb{D})$ is significantly larger than $A(\mathbb{D})$. Indeed, in Section 7, we assert that $A(\mathbb{D}) \subsetneqq Z^\infty(\mathbb{D})$. However, we do not know the measure of the set $H^\infty(\mathbb{D}) \setminus Z^\infty(\mathbb{D})$. Nevertheless, in the category of functions in $Z^\infty(\mathbb{D})$, outer functions represent singly generated M-ideals in $H^\infty(\mathbb{D})$ (see Theorem 6.2):

Theorem 1.11. *Let $f \in Z^\infty(\mathbb{D})$. Then $I_{H^\infty(\mathbb{D})}(f)$ is an M-ideal in $H^\infty(\mathbb{D})$ if and only if f is an outer function.*

Moreover, in the case of outer functions in $Z^\infty(\mathbb{D})$, the corresponding singly generated M-ideals are explicit: If $f \in Z^\infty(\mathbb{D})$ is an outer function, then (see Corollary 7.1)

$$\mathcal{I}(f) = \{g \in H^\infty(\mathbb{D}) : Z_{\mathbb{T}}(f) \subseteq Z_{\mathbb{T}}(g) \text{ and } g \text{ has a continuous extension to } Z_{\mathbb{T}}(f)\}.$$

In Example 7.6, we present an M-ideal $I_{H^\infty(\mathbb{D})}(f)$ in $H^\infty(\mathbb{D})$ such that f is an outer function and

$$f \notin Z^\infty(\mathbb{D}).$$

All the results and examples presented here suggest that the structure of M-ideals in $H^\infty(\mathbb{D})$ is by far complicated, even at the level of singly generated ideals in $H^\infty(\mathbb{D})$.

We have also revealed the structure of M-ideals in $H^\infty(\mathbb{D})$ that are finitely generated by functions from $Z^\infty(\mathbb{D})$. Given $\{f_1, \ldots, f_m\} \subseteq Z^\infty(\mathbb{D})$, denote by $I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m)$ the closed ideal generated by $\{f_1, \ldots, f_m\}$. In Theorem 8.1, we prove:

Theorem 1.12. *Let $\{f_1, \ldots, f_m\} \subseteq Z^\infty(\mathbb{D})$. If $I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m)$ is an M-ideal in $H^\infty(\mathbb{D})$, then there exists an outer function $f \in Z^\infty(\mathbb{D})$ such that*

$$I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m) = I_{H^\infty(\mathbb{D})}(f).$$

The above result has some peculiarities within the framework of ideal theory. Indeed, this is reminiscent of the ascending chain condition in a ring. More specifically, the finitely generated ideal subjected to the M-ideal condition yields the subsequent terminating fact:

$$I_{H^\infty(\mathbb{D})}(f_1) \subseteq I_{H^\infty(\mathbb{D})}(f_1, f_2) \subseteq \cdots \subseteq I_{H^\infty(\mathbb{D})}(f_1, \cdots, f_m) = I_{H^\infty(\mathbb{D})}(f).$$

We further recall that a ring R satisfies the ascending chain condition on ideals if and only if every ideal of R is finitely generated. Note that rings that satisfy the ascending chain condition on ideals are also called *Noetherian*.

Alongside the results mentioned above, this article covers additional results about M-ideals, p-sets, and associated areas. Furthermore, an ample number of examples and counterexamples have been provided. We refer the reader to [6] for the theory of M-ideals in the context of Banach algebras, such as M-ideals that are ideals. Also see [16] for some natural examples of M-ideals, and see [39] for more exotic examples of M-ideals.

We shall now delineate the structure of the subsequent sections of the paper. Section 2 provides an overview of the foundational concepts, recalls definitions, and conveys known results that will be used in the next sections. Section 3 illustrates the extensive magnitude of M-ideals in $H^\infty(\mathbb{D})$, while Section 4 discusses p-sets from the standpoint of the space $H^\infty(\mathbb{D})$. In Section 5, we introduce the notion of analytic primes in $H^\infty(\mathbb{D})$ and prove that proper M-ideals are analytic primes. A number of the results presented in this section pertain to Hilbert function space theory, such as Toeplitz operators, outer functions, and inner functions. Each result in this section applies to several variables. The primary emphasis of Section 6 is on principal ideals in $H^\infty(\mathbb{D})$ that are generated by functions from $Z^\infty(\mathbb{D})$. Section 7 provides direct applications of results from previous sections and offers examples that illustrate the intricacy of the functions introduced earlier. Section 8 focuses on M-ideals generated by finite number functions in $Z^\infty(\mathbb{D})$.

2. Preliminaries

We already pointed out that $H^\infty(\mathbb{D})$ is a uniform algebra. Here we begin with the formal definition of uniform algebras. A *uniform algebra* on a compact Hausdroff space K is a uniformly closed subalgebra of $C(K)$ which contains the constant functions and separates the points of K. Evidently, a uniform algebra is a commutative Banach algebra (under the uniform norm).

Now we set up some basic structures of commutative Banach algebras. Let A be a commutative Banach algebra with unit. Let us denote by $\mathcal{M}(A)$ the maximal ideal space of A. Since A is unital, we know that $\mathcal{M}(A) \subseteq S_{A^*}$, where given a Banach space X, we define the unit sphere of X as

$$S_X = \{x \in X : \|x\|_X = 1\}.$$

Hence by the Banach–Alaoglu theorem and the fact that $\mathcal{M}(A)$ is closed, it follows that $\mathcal{M}(A)$ is a compact Hausdorff space in the weak-$*$ topology. If we denote by $C(\mathcal{M}(A))$ the algebra of continuous functions on $\mathcal{M}(A)$ with the supremum norm, then the *Gelfand map*

$$\Gamma : A \to C(\mathcal{M}(A)),$$

is a contractive homomorphism of Banach algebras, where

$$\Gamma(f) = \hat{f},$$

and

$$\hat{f}(\varphi) = \varphi(f),$$

for all $f \in A$ and $\varphi \in \mathcal{M}(A)$. A subset $C \subseteq K$ is called a *boundary* for A if

$$\sup_{x\in K}|f(x)| = \max_{x\in C}|f(x)|,$$

for all $f \in A$. The *Šilov boundary* for A, denoted by $\partial_S A$, is the smallest closed boundary for A, that is [25, page 173]

$$\partial_S A = \bigcap\{C \subseteq K : C \text{ is a closed boundary for } A\}.$$

Equivalently, $\partial_S A$ is the smallest closed subset of $\mathcal{M}(A)$ on which every $\hat{f} \in \Gamma(A)$ attains its maximum modulus. Next, we turn to the unital commutative Banach algebra $H^\infty(\mathbb{D})$ [25, Chapter 10]. In this case

$$\|f\|_\infty = \|\hat{f}\|_\infty = \sup_{\varphi\in\mathcal{M}(H^\infty(\mathbb{D}))}|\hat{f}(\varphi)|, \tag{2.1}$$

for all $f \in H^\infty(\mathbb{D})$, that is, Γ an isometric isomorphism from $H^\infty(\mathbb{D})$ into $C(\mathcal{M}(H^\infty(\mathbb{D})))$. Evidently, Γ is not onto. We use the Gelfand map $\Gamma : H^\infty(\mathbb{D}) \to C(\mathcal{M}(H^\infty(\mathbb{D})))$ to identify $H^\infty(\mathbb{D})$ with $\widehat{H^\infty(\mathbb{D})}$, where

$$\widehat{H^\infty(\mathbb{D})} := \Gamma(H^\infty(\mathbb{D})),$$

Therefore, $\widehat{H^\infty(\mathbb{D})}$, and hence an isometric copy of $H^\infty(\mathbb{D})$, is a uniformly closed subalgebra of $C(\mathcal{M}(H^\infty(\mathbb{D})))$. In particular, $\widehat{H^\infty(\mathbb{D})}$ contains constant functions and separates points [25].

Recall from (1.1) that for a function $f \in H^\infty(\mathbb{D})$, the radial limit extension of f is denoted by $\tilde{f}$, where

$$\tilde{f}(z) = \lim_{r\to 1^-} f(rz) \qquad (z \in \mathbb{T} \textit{ a.e.}).$$

This yields a natural way to identify $H^\infty(\mathbb{D})$ with a closed subalgebra of $L^\infty(\mathbb{T})$, where $L^\infty(\mathbb{T})$ denotes the von Neumann algebra of all essentially bounded measurable complex-valued functions on $\mathbb{T}$. If $\widetilde{H^\infty(\mathbb{T})}$ denotes the copy of $H^\infty(\mathbb{D})$ in $L^\infty(\mathbb{T})$, then

$$\widetilde{H^\infty(\mathbb{T})} = L^\infty(\mathbb{T}) \cap \widetilde{H^2(\mathbb{T})},$$

where $\widetilde{H^2(\mathbb{T})} \subseteq L^2(\mathbb{T})$ is the Hardy space on the unit circle $\mathbb{T}$ (again, via radial limits as in (1.1)). In view of the above identification, we denote by $\tilde{f} \in \widetilde{H^\infty(\mathbb{T})}$ the function corresponding to $f \in H^\infty(\mathbb{D})$. In other words, we can represent $H^\infty(\mathbb{D})$ as a uniformly closed subalgebra of $L^\infty(\mathbb{T})$. This point of view is useful in identifying the Šilov boundary of $H^\infty(\mathbb{D})$ in the sense that

$$\partial_S H^\infty(\mathbb{D}) = \tau(\mathcal{M}(L^\infty(\mathbb{T}))), \tag{2.2}$$

where the map $\tau : \mathcal{M}(L^\infty(\mathbb{T})) \to \mathcal{M}(H^\infty(\mathbb{D}))$ defined by

$$\tau(\varphi) = \varphi|_{H^\infty(\mathbb{D})} \qquad (\varphi \in \mathcal{M}(L^\infty(\mathbb{T}))),$$

is a homeomorphism [25, page 174]. These tools will frequently be implemented in the next sections. In addition, we will see soon that p-sets are closely connected to the theory of M-ideals in $H^\infty(\mathbb{D})$. We recall:

Definition 2.1. *Let A be a uniform algebra on a compact Hausdorff space K. A closed subset $C \subseteq K$ is said to be a peak set if there exists a function $f \in A$ such that*

$$f|_C \equiv 1,$$

and

$$|f(x)| < 1 \qquad (x \in X \setminus C).$$

We frequently say that f peaks on C.

Clearly, if $f \in A$ peaks on C, then

$$\|f\| = 1.$$

It is easy to see that a countable intersection of peak sets is a peak set [15, page 208]. In general, we define:

Definition 2.2. *Let A be a uniform algebra on a compact Hausdorff space K. A set $P \subset K$ is called a p-set of A if P is the intersection of a family of peak sets.*

Some of the well-known and general features of p-sets are as follows: (i) A p-set is a peak set if and only if it is a G_δ-set [13, Lemma 12.1]. (ii) A countable union of p-sets is a p-set whenever the union is a closed set [13, Corollary 12.8].

We conclude this introductory section by recalling the following characterizations of M-ideals in uniform algebras in terms of p-sets and bounded approximate units. This result will be frequently used in what follows.

Theorem 2.3. [18, Chapter V, Theorem 4.2] *Let A be a uniform algebra and let J a closed subspace of A. Then the following conditions are equivalent:*

(i) *J is an M-ideal of A.*
(ii) *J is the annihilator of a p-set of A.*
(iii) *J is an ideal of A containing a bounded approximate unit.*

Recall that a bounded approximate unit in a commutative Banach algebra A is a bounded net $\{x_i\}_{i\in\Lambda} \subseteq A$ such that

$$\|x_i x - x\|_A \to 0 \qquad (x \in A),$$

along the net. In our analysis, we will mostly use the equivalence of (i) and (iii) in the preceding theorem. More specifically, we will enhance the practical relevance of this criterion by applying it to the specific context of uniform algebra $H^\infty(\mathbb{D})$.

3. M-ideals are colossal

The maximal ideal space of $H^\infty(\mathbb{D})$ is renowned for its vastness and complex peculiarities. Put simply, the maximal ideal space of $H^\infty(\mathbb{D})$ is highly intricate and serves as a hindrance to fully understanding the structure and characteristics of $H^\infty(\mathbb{D})$. The objective of this brief section is to highlight the notoriety of the space of M-ideals, just analogous to the maximal ideal space of $H^\infty(\mathbb{D})$. First, we recall a lemma concerning classification of p-sets [18, V, Lemma 4.3]. Denote by e the unit of the given uniform algebra.

Lemma 3.1. *Let A be an uniform algebra on a compact Hausdorff space K, and let D be a closed subset of K. Then D is a p-set for A if and only if for each $\varepsilon > 0$ and open set $U \supseteq D$, there exists $a \in A$ such that the following three conditions hold:*

1. $\|e - a\| \leq 1 + \varepsilon$,
2. $a|_D = 0$, *and*
3. $\left|(e - a)|_{K\setminus U}\right| < \varepsilon$.

We also need to recall a Urysohn-type lemma for $H^\infty(\mathbb{D})$ from [4, Lemma 2.1].

Lemma 3.2. *Let U be an open subset of $\mathcal{M}(H^\infty(\mathbb{D}))$. Then for each $\varepsilon \in (0,1)$ and $\varphi_0 \in U \cap \partial_S H^\infty(\mathbb{D})$, there exists $f \in H^\infty(\mathbb{D})$ such that*

1. $\|\hat{f}\| = \hat{f}\,(\varphi_0) = 1$,
2. $\sup\left\{|\hat{f}(\varphi)| : \varphi \in \partial_S H^\infty(\mathbb{D})\setminus U\right\} < \varepsilon$, *and*
3. $|\hat{f}(\varphi)| + (1-\varepsilon)|1 - \hat{f}(\varphi)| \leq 1$ *for all* $\varphi \in \mathcal{M}(H^\infty(\mathbb{D}))$.

Now we turn to M-ideals in $H^\infty(\mathbb{D})$. Fix $\psi \in \partial_S H^\infty(\mathbb{D})$ and $\varepsilon > 0$. Then there exists a function $f \in H^\infty(\mathbb{D})$ such that f satisfies (1) to (3) of Lemma 3.2. If we let

$$g = 1 - f,$$

then

$$g|_{\{\psi\}} = 0,$$

and

$$\|1 - g\| = 1,$$

and

$$|(1-g)|_{\partial_S H^\infty(\mathbb{D})\setminus U}\| < \varepsilon.$$

This fulfils all requirements of Lemma 3.1, with e equaling 1 and a equaling g. Consequently, it follows that $\{\psi\}$ is a p-set of $H^\infty(\mathbb{D})$. Hence, by Theorem 2.3, $\ker\psi$ is an M-ideal in $H^\infty(\mathbb{D})$. Consequently, we have established the following result, which, in particular, states that the collection of M-ideals in $H^\infty(\mathbb{D})$ is immense.

Proposition 3.3. *Each maximum ideal in $H^\infty(\mathbb{D})$ that corresponds to a complex homomorphism in $\partial_S H^\infty(\mathbb{D})$ is an M-ideal in $H^\infty(\mathbb{D})$.*

On the contrary, it is easy to verify that each closed subset of $\mathbb{T}$ is a p-set for $C(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of all continuous functions on the unit circle $\mathbb{T}$. As a result, M-ideals in $C(\mathbb{T})$ are ideals of functions that vanish on some closed subset of $\mathbb{T}$. Observe that $C(\mathbb{T})$ is a commutative C^*-algebra.

On the other hand, the disc algebra $A(\mathbb{D})$ is a commutative Banach algebra that is also a uniform algebra on $\overline{\mathbb{D}}$. At this point, we recall the Glicksberg peak set theorem [13, p. 58]:

Let A be a uniform algebra on a compact Hausdorff space K, and let D be a closed subset of K. Then D is a p-set if and only if $\mu_D \in A^\perp$ for all measure $\mu \in A^\perp$. That is, D is a p-set if and only if $\mu|_E$ is orthogonal to A for all measures μ orthogonal to A.

Returning to the uniform algebra $A(\mathbb{D})$, we first observe in view of the F. and M. Reisz theorem, that any measure orthogonal to $A(\mathbb{D})$ is absolutely continuous with respect to the Lebesgue measure. Hence the p-sets of $A(\mathbb{D})$ are precisely the closed subsets of $\mathbb{T}$ having Lebesgue measure zero. Therefore, a closed subspace $C \subseteq A(\mathbb{D})$ is an M-ideals in $A(\mathbb{D})$ if and only if there exits a subset $D \subset \mathbb{T}$ of Lebesgue measure 0 such that

$$C = \{f \in A(\mathbb{D}) : f|_D = 0\}.$$

In the context of M-ideals in vector-valued disc algebras, we refer the reader to [1]. Clearly, the structure of M-ideals in $A(\mathbb{D})$ is simple and clean. On the contrary, the above proposition clearly suggests that the situation of representing M-ideals in $H^\infty(\mathbb{D})$ is a complex problem. Our results and methodology in this paper will also emphasise this characteristic.

4. p-sets

According to Theorem 2.3, the investigation of M-ideals in $H^\infty(\mathbb{D})$ is comparable to the analysis of p-sets. The theory of p-sets presents an equally challenging problem for the space $H^\infty(\mathbb{D})$, just like M-ideals. Within this section, our objective is to pick up a few significant characteristics of p-sets, specifically focusing on concrete examples of M-ideals in our particular scenario. Naturally, the results also hold independent significance for the scrutiny of p-sets of $H^\infty(\mathbb{D})$.

We start with representations of peak sets of $H^\infty(\mathbb{D})$. For each $\alpha \in \mathbb{T}$ and $f \in S_{H^\infty(\mathbb{D})}$, we define

$$\mathscr{M}_\alpha^f = \{\varphi \in \mathscr{M}(H^\infty(\mathbb{D})) : \hat{f}(\varphi) = \alpha\}.$$

It now follows from the definition itself that the peak sets of $H^\infty(\mathbb{D})$ admit the form $\mathscr{M}_\alpha^f$ for some $\alpha \in \mathbb{T}$ and $f \in S_{H^\infty(\mathbb{D})}$. If $f = z \in S_{H^\infty(\mathbb{D})}$, then we simply write

$$\mathscr{M}_\alpha = \mathscr{M}_\alpha^f \qquad (\alpha \in \mathbb{T}).$$

Therefore, some simple examples of peak sets in $H^\infty(\mathbb{D})$ includes

$$\mathscr{M}_\alpha = \{\varphi \in \mathcal{M}(H^\infty(\mathbb{D})) : \hat{z}(\varphi) = \alpha\} \qquad (\alpha \in \mathbb{T}).$$

Now we turn to p-sets and prove that a p-set of $H^\infty(\mathbb{D})$ must meet the Šilov boundary $\partial_S H^\infty(\mathbb{D})$.

Theorem 4.1. *Let Q be a nonempty subset of $\mathcal{M}(H^\infty(\mathbb{D})) \setminus \partial_S H^\infty(\mathbb{D})$. Then Q is not a p-set of $H^\infty(\mathbb{D})$.*

Proof. Let Q is a non-empty subset of $\mathcal{M}(H^\infty(\mathbb{D}))$. Let

$$P = Q \cap \partial_S H^\infty(\mathbb{D}).$$

Assume that Q is p-set of $H^\infty(\mathbb{D})$. We claim that $P \neq \emptyset$. To this end, suppose $\{f_i\}_{i\in I} \subseteq S_{H^\infty(\mathbb{D})}$ is the family of functions corresponding to the p-set Q, that is

$$Q = \bigcap_{i\in I} \mathscr{M}_1^{f_i}.$$

Since $P = Q \cap \partial_S H^\infty(\mathbb{D})$, we have

$$P = \bigcap_{i\in I} \mathscr{M}_1^{f_i} \bigcap \partial_S H^\infty(\mathbb{D}).$$

Observe that $\mathcal{M}(H^\infty(\mathbb{D}))$ is compact and $\mathscr{M}_1^{f_i}$ is closed for each i. Therefore, if the collection

$$\{\mathscr{M}_1^{f_i} \cap \partial_S H^\infty(\mathbb{D})\}_{i\in I},$$

has finite intersection property (FIP), then one can conclude that $P \neq \emptyset$. To prove the FIP, for each finite collection $\{f_1, \dots, f_n\} \subset S_{H^\infty(\mathbb{D})}$, define

$$\mathscr{M}^{f_{i_1}, f_{i_2}, \dots, f_{i_n}} = \bigcap_{j=1}^{n} \mathscr{M}_1^{f_{i_j}}.$$

As $\mathscr{M}^{f_{i_1}, f_{i_2}, \dots, f_{i_n}}$ is a finite intersection of peak sets, there exists a function $g \in S_{H^\infty(\mathbb{D})}$ such that

$$\mathscr{M}_1^{g} = \mathscr{M}^{f_{i_1}, f_{i_2}, \dots, f_{i_n}}.$$

Since $\|g\| = 1$ and $\partial_S H^\infty(\mathbb{D})$ is the Šilov boundary of $H^\infty(\mathbb{D})$, there is at least one $\psi \in \partial_S H^\infty(\mathbb{D})$ such that

$$\psi(g) = 1.$$

This clearly implies that

$$\psi \in \mathscr{M}_1^{g} \cap \partial_S H^\infty(\mathbb{D}),$$

and hence

$$\begin{aligned} \mathscr{M}_1^{g} \cap \partial_S H^\infty(\mathbb{D}) &= \mathscr{M}^{f_{i_1}, f_{i_2}, \dots, f_{i_n}} \cap \partial_S H^\infty \\ &= \cap_{j=1}^{n} \mathscr{M}_1^{f_{i_j}} \cap \partial_S H^\infty(\mathbb{D}) \\ &\neq \emptyset. \end{aligned}$$

Therefore, the collection $\{\mathscr{M}_1^{f_i} \cap \partial_S H^\infty(\mathbb{D})\}_{i\in I}$ satisfies the FIP, from which one concludes that $P \neq \emptyset$. $\square$

Therefore, we have the following: If Q is a p-set of $H^\infty(\mathbb{D})$, then

$$Q \cap \partial_S H^\infty(\mathbb{D}) \neq \emptyset.$$

For each $\alpha \in \mathbb{D}$, denote by ev_α the evaluation functional at α, that is

$$ev_\alpha(f) = f(\alpha) \qquad (f \in H^\infty(\mathbb{D})).$$

The following is a consequence of Theorem 4.1:

Corollary 4.2. *Let $\alpha \in \mathbb{D}$. Then $\{\alpha\}$ is not a p-set of $H^\infty(\mathbb{D})$. In particular, the maximal ideal* $\ker ev_\alpha$ *is not an M-ideal in $H^\infty(\mathbb{D})$.*

Therefore, any subset of the unit disc $\mathbb{D}$ is not a p-set of $H^\infty(\mathbb{D})$. On the other hand, the equivalence of (i) and (ii) in Theorem 2.3 says that the M-ideals in a uniform algebra A are precisely of the form

$$J_P = \{f \in A : f|_P = 0\}, \tag{4.1}$$

where $P \subseteq K$ is a p-set of A. In general, we use J_P to denote the set of all functions from the uniform algebra under consideration that vanish on a set $P \subseteq K$.

Typically, M-ideals in Banach spaces do not have a direct correlation (but some analogy; see the remark in the first paragraph of Section 1) with the notion of ideals in rings. Nevertheless, the above representations of M-ideals promptly reveals that this is not the case for uniform algebras.

Corollary 4.3. *Let A be a uniform algebra. Then M-ideals in A are ideals in the ring A.*

Considering an M-ideal J in $H^\infty(\mathbb{D})$, we will now establish a connection with two possible p-set representations of $H^\infty(\mathbb{D})$ corresponding to J. Establishing such a link is inescapable from the perspective of two representations of the uniform algebra $H^\infty(\mathbb{D})$ over the compact sets $\mathcal{M}(H^\infty(\mathbb{D}))$ and $\partial_S H^\infty(\mathbb{D})$. To be more precise, if J is an M-ideal in $H^\infty(\mathbb{D})$, then according to the representation (4.1), there are two subsets $P \subseteq \mathcal{M}(H^\infty(\mathbb{D}))$ and $Q \subseteq \partial_S H^\infty(\mathbb{D})$ such that

$$J = J_P = J_Q.$$

In the following, we aim to establish a correlation between P and Q. Part of the proof follows the lines of Hoffman [25, page 187].

Theorem 4.4. *Let J be an M-ideal in $H^\infty(\mathbb{D})$. Suppose*

$$J = J_P,$$

where $P \subseteq \mathcal{M}(H^\infty(\mathbb{D}))$ is the corresponding p-set. Then

$$J = J_{P \cap \partial_S H^\infty(\mathbb{D})}.$$

Proof. Given the representing p-set $P \subseteq \mathcal{M}(H^\infty(\mathbb{D}))$ of the M-ideal J, we define

$$Q = P \cap \partial_S H^\infty(\mathbb{D}).$$

By Theorem 4.1, we know that $Q \neq \emptyset$. Let $\{f_i\}_{i\in I} \subseteq S_{H^\infty(\mathbb{D})}$ be the family of functions corresponding to the p-set P. Then (see the notation preceding Theorem 4.1)

$$P = \bigcap_{i \in I} \mathscr{M}_1^{f_i}.$$

Fix an $i \in I$, and pick $\psi \in \mathcal{M}_1^{f_i}$. Denote by m_ψ the unique representing measure of ψ on $\partial_S H^\infty(\mathbb{D})$. We claim that m_ψ is supported on $\mathcal{M}_1^{f_i} \cap \partial_S H^\infty(\mathbb{D})$. To see this, first we set

$$h = \frac{1 + f_i}{2}.$$

For each $n \geq 1$, we observe that

$$\begin{aligned} \int_{\partial_S H^\infty(\mathbb{D})} \hat{h}^n dm_\psi &= \psi(h^n) \\ &= 1. \end{aligned}$$

On the other hand, the sequence of functions $\{\hat{h}^n\}_{n\geq 1}$ is bounded and

$$\hat{h}^n \longrightarrow \chi_{\mathcal{M}_1^{f_i}},$$

pointwise, where $\chi_{\mathcal{M}_1^{f_i}}$ denotes the indicator function of $\mathcal{M}_1^{f_i}$. By the dominated convergence theorem, we have

$$1 = \lim_n \int_{\partial_S H^\infty(\mathbb{D})} \hat{h}^n dm_\psi = \int_{\partial_S H^\infty(\mathbb{D})} \hat{\chi}_{\mathcal{M}_1^{f_i}} dm_\psi = \int_{\mathcal{M}_1^{f_i} \cap \partial_S H^\infty(\mathbb{D})} dm_\psi,$$

and hence the measure m_ψ is supported on $\mathcal{M}_1^{f_i} \cap \partial_S H^\infty(\mathbb{D})$, completing the proof of the claim. Since $P = \bigcap_{i\in I} \mathcal{M}_1^{f_i}$, it follows that each ψ in P is supported on $\mathcal{M}_1^{f_i} \cap \partial_S H^\infty(\mathbb{D})$ for all $i \in I$. Hence the minimal support of ψ is contained in

$$\bigcap_{i\in I} \left(\mathcal{M}_1^{f_i} \cap \partial_S H^\infty(\mathbb{D})\right) = P \cap \partial_S H^\infty(\mathbb{D}) = Q.$$

Therefore, for any $f \in J_{P\cap\partial_S H^\infty(\mathbb{D})} = J_Q$ and $\psi \in P$, we have

$$\psi(f) = \int_{P\cap\partial_S H^\infty(\mathbb{D})} \hat{f} dm_\psi = 0,$$

and hence $J_Q \subseteq J_P$. Since $Q \subseteq P$, it also follows that $J_Q \subseteq J_P$, and consequently

$$J_P = J_Q,$$

which completes the proof of the theorem. □

5. Analytic primes

This section will emphasise some significant features of M-ideals in $H^\infty(\mathbb{D})$. The properties of M-ideals that are being addressed here are also applicable to the polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, $n \geq 1$. Now, the definition of $H^\infty(\mathbb{D}^n)$ is comparable to that of the single variable:

$$H^\infty(\mathbb{D}^n) = \{f \in \mathrm{Hol}(\mathbb{D}^n) : \|f\| := \sup_{z\in\mathbb{D}^n} |f(z)| < \infty\}.$$

By the same argumentation, $H^\infty(\mathbb{D}^n)$ is also a uniform algebra, and so Theorem 2.3 applies to $H^\infty(\mathbb{D}^n)$.

We stated at the very beginning of Section 1 that the notion of M-ideals was proposed in [3] as a generalization of two-sided ideals in Banach spaces. Our key result in this section is yet another algebraic property of M-ideals. Recall that an ideal P in a commutative ring R is called prime if $P \neq R$ and if a and b are two elements of R such that

$$ab \in P,$$

then either $a \in P$ or $b \in P$. This motivates the following analytic definition of prime ideals in $H^\infty(\mathbb{D}^n)$:

Definition 5.1. *Let J be a proper nontrivial closed subspace of $H^\infty(\mathbb{D}^n)$. We call J an analytic prime as long as the following property holds true: If f and g are two elements of $H^\infty(\mathbb{D}^n)$ such that*

$$fg \in J,$$

and

$$|\tilde{f}(z)| > \delta \qquad (z \in \mathbb{T}^n),$$

for some $\delta > 0$, then

$$g \in J.$$

Note that, just as in the single variable case, here also a function $f \in H^\infty(\mathbb{D}^n)$ admits a boundary value $\tilde{f}$ a.e. on the n-torus $\mathbb{T}^n$ [40]. It ought to be observed that the notion of analytic primes has the potential to extend to a broad uniform algebra (one perhaps requires finding a suitable replacement for $\mathbb{T}^n$). It is also clear that the concept of an analytic prime is more potent than prime ideals in $H^\infty(\mathbb{D}^n)$ (or a uniform algebra). This also pertains to the structure of M-ideals in $H^\infty(\mathbb{D}^n)$. Specifically:

Theorem 5.2. *Proper nontrivial M-ideals in $H^\infty(\mathbb{D}^n)$ are analytic primes.*

Proof. Let J be a proper nontrivial M-ideal in $H^\infty(\mathbb{D}^n)$. Since J is an M-ideal and $H^\infty(\mathbb{D}^n)$ is a uniform algebra, it follows that J contains a bounded approximate unit $\{f_\lambda\}_{\lambda\in\Lambda}$. Let $f, g \in H^\infty(\mathbb{D}^n)$, and suppose $fg \in J$. Suppose there is a $\delta > 0$ such that

$$|\tilde{f}(z)| > \delta \qquad (z \in \mathbb{T}^n).$$

Fix $\varepsilon > 0$. There exists $\lambda_\varepsilon \in \Lambda$ such that

$$\|fgf_\lambda - fg\| \leq \varepsilon,$$

for all $\lambda \geq \lambda_\varepsilon$. Since this is the case, there exists a sequence $\{\lambda_m\} \subseteq \Lambda$ such that

$$\|fgf_{\lambda_m} - fg\| \leq \frac{1}{m},$$

for all $m \geq 1$. In view of the identification of functions via radial limits, we now observe that, for functions in $H^\infty(\mathbb{D}^n)$, the supremum on the boundary is sufficient to take into account. Section 2 describes the identification for this $n = 1$ scenario, whereas the $n > 1$ case works similarly. Fix an integer $m \geq 1$. For all $z \in \mathbb{T}^n$, we have

$$\begin{aligned}\delta|\tilde{g}(z)\tilde{f}_{\lambda_m}(z) - \tilde{g}(z)| &\leq |\tilde{f}(z)||\tilde{g}(z)\tilde{f}_{\lambda_m}(z) - \tilde{g}(z)|\\ &= |\tilde{f}(z)\tilde{g}(z)\tilde{f}_{\lambda_m}(z) - \tilde{f}(z)\tilde{g}(z)|.\end{aligned}$$

Taking the supremum over all $z \in \mathbb{T}^n$ gives

$$\delta\|\tilde{g}\tilde{f}_{\lambda_m} - \tilde{g}\| \leq ||\tilde{f}\tilde{g}\tilde{f}_{\lambda_m} - \tilde{f}\tilde{g}||,$$

and hence

$$\|\tilde{g}\tilde{f}_{\lambda_m} - \tilde{g}\| \leq \frac{1}{\delta}\frac{1}{m},$$

for all $m \geq 1$. Since

$$\tilde{g}\tilde{f}_{\lambda_m} \in J \qquad (m \geq 1),$$

it follows that $g \in J$. Thus, the proof of the theorem is concluded. $\square$

The analytic prime property of M-ideals in $H^\infty(\mathbb{D})$ will be frequently used in what follows. We now present one implication from the preceding result, which is of independent relevance in terms of the theory of bounded analytic functions and the theory of M-ideals. Recall that a function $\upsilon \in H^\infty(\mathbb{D}^n)$ is said to be *inner* if

$$|\tilde{\upsilon}(z)| = 1,$$

for all $z \in \mathbb{T}^n$ a.e. Now we prove that an M-ideal never intersects the multiplicative set of inner functions.

Corollary 5.3. *Let J be a nontrivial and proper M-ideal in $H^\infty(\mathbb{D}^n)$. Then J does not have any inner functions.*

Proof. Recall from Corollary 4.3 that J is a (two sided) ideal of $H^\infty(\mathbb{D}^n)$. Suppose that there exists an inner function $\upsilon \in H^\infty(\mathbb{D}^n)$ such that $\upsilon \in J$. Then, by writing

$$\upsilon = 1 \times \upsilon,$$

we conclude from Theorem 5.2 that $1 \in J$. This is contrary to the fact that J is proper. $\square$

There is an alternate proof of the above, which is an application of Theorem 4.1 and representations of M-ideals in uniform algebras. Recall from (4.1) that there exists $P \subseteq \mathcal{M}(H^\infty(\mathbb{D}^n))$ such that $J = J_P$. Now Theorem 4.1 implies that P must meet the Šilov boundary of $H^\infty(\mathbb{D}^n)$. However, inner functions do not vanish on the Šilov boundary [25, page 179].

Recall that singular inner functions are those inner functions that are zero-free. As a visual, for each $\alpha \in \mathbb{T}$, the function

$$\upsilon_\alpha(z) := \exp\Big(\frac{z+\alpha}{z-\alpha}\Big) \qquad (z \in \mathbb{D}),$$

is a singular inner function in $H^\infty(\mathbb{D})$. We illustrate the above result through the following example:

Example 5.4. *Consider the closed ideal J in $H^\infty(\mathbb{D})$, where*

$$J = \{f \in H^\infty(\mathbb{D}) : \lim_{r\to 1} f(r) = 0\}.$$

It is easy to see that $\upsilon_1 \in J$. Hence, by Corollary 5.3, it follows that J is not an M-ideal in $H^\infty(\mathbb{D})$.

Now we present another application of Theorem 5.2. This time, it will be applied to the Hardy space $H^2(\mathbb{D}^n)$. Recall that [40]

$$H^2(\mathbb{D}^n) = \Big\{f \in \mathrm{Hol}(\mathbb{D}^n) : \|f\|_2 := \Big(\sup_{0<r<1} \int_{\mathbb{T}^n} |f(rz)|^2 d\mu(z)\Big)^{\frac{1}{2}} < \infty\Big\},$$

where $d\mu$ denotes the normalized Lebesgue measure on $\mathbb{T}^n$ and $rz = (rz_1, \ldots, z_n)$. In view of the radial limits, we have the following isometric embedding:

$$H^2(\mathbb{D}^n) \hookrightarrow L^2(\mathbb{T}^n).$$

Moreover, it is well-known that

$$\overline{\mathbb{C}[z_1, \ldots, z_n]}^{H^2(\mathbb{D}^n)} = H^2(\mathbb{D}^n). \tag{5.1}$$

We also need some operator theoretic concepts that are classic in nature. Given a function $\varphi \in H^\infty(\mathbb{D}^n)$, we define T_φ acting on $H^2(\mathbb{D}^n)$ by

$$T_\varphi f = \varphi f \qquad (f \in H^2(\mathbb{D}^n)).$$

We call T_φ the *Toeplitz operator* with the symbol φ. To follow the convention, we choose to abuse the notation here; that is, T_φ instead of T_f. The symbols with the independent variables $\{z_1, \ldots, z_n\}$ are known as distinguished Toeplitz operators. In this case, we have

$$T_{z_i} f = z_i f \qquad (f \in H^2(\mathbb{D}^n)),$$

for all $i = 1, \ldots, n$. We now show that the density assertion in (5.1) is true for each M-ideal in $H^\infty(\mathbb{D}^n)$.

Theorem 5.5. *If J is a nontrivial M-ideal in $H^\infty(\mathbb{D}^n)$, then*

$$\overline{J}^{H^2(\mathbb{D}^n)} = H^2(\mathbb{D}^n).$$

Proof. We know, by (4.1), that

$$J = \{\varphi \in H^\infty(\mathbb{D}) : \varphi|_P = 0\},$$

for some p-set of $H^\infty(\mathbb{D}^n)$. From here, it is easy to conclude that J is invariant under z_i, $i = 1, \ldots, n$. Therefore, $\overline{J}^{H^2(\mathbb{D}^n)}$ is also invariant under z_i, $i = 1, \ldots, n$. It is then sufficient to show that 1 is in $\overline{J}^{H^2(\mathbb{D}^n)}$. To accomplish this, we again use the bounded approximate unit of M-ideals as we did in the proof of Theorem 5.2. Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a bounded approximate unit in J. As in the proof of Theorem 5.2, for a fixed $\varphi \in J$, there exists a sequence $\{\lambda_m\} \subseteq \Lambda$ such that

$$\|\varphi\varphi_{\lambda_m} - \varphi\|_{H^\infty(\mathbb{D}^n)} \leq \frac{1}{m}, \tag{5.2}$$

for all $m \geq 1$. Note that $\{\varphi_{\lambda_m}\}$ is a bounded sequence in $H^2(\mathbb{D}^n)$. Consequently, $\{\varphi_{\lambda_m}\}$ has a weak convergent subsequence, and hence, there exists $f \in H^2(\mathbb{D}^n)$ such that

$$\varphi_{\lambda_{m_k}} \xrightarrow{w} f,$$

in $H^2(\mathbb{D}^n)$. Since the Toeplitz operator T_φ on $H^2(\mathbb{D}^n)$ is bounded, we obtain

$$\varphi\varphi_{\lambda_{m_k}} \xrightarrow{w} \varphi f,$$

in $H^2(\mathbb{D}^n)$. Also by (5.2), we have

$$\varphi\varphi_{\lambda_{m_k}} \xrightarrow{w} \varphi,$$

in $H^2(\mathbb{D}^n)$. Here we are using the general fact that $H^\infty(\mathbb{D}^n)$ is contractively embedded in $H^2(\mathbb{D}^n)$. Therefore

$$\varphi f = \varphi,$$

equivalently

$$f - 1 \in \ker T_\varphi.$$

However, since φ is an analytic function, we know that

$$\ker T_\varphi = \{0\}.$$

This effortlessly implies

$$f = 1,$$

and completes the proof of the theorem, as weak closure and norm closure are identical for convex sets. □

Recall that for a given $f \in H^\infty(\mathbb{D}^n)$, we denote by $I_{H^\infty(\mathbb{D}^n)}(f)$ the closed principal ideal generated by f in $H^\infty(\mathbb{D}^n)$ (see(1.2)):

$$I_{H^\infty(\mathbb{D}^n)}(f) = \overline{\{pf : p \in \mathbb{C}[z_1, \ldots, z_n]\}}^{H^\infty(\mathbb{D}^n)}.$$

The following is a straight application of the above theorem, which establishes a clear link between M-ideals and classical operators such as Toeplitz operators.

Corollary 5.6. *Let $\varphi \in H^\infty(\mathbb{D}^n)$. If $I_{H^\infty(\mathbb{D}^n)}(\varphi)$ is an M-ideal in $H^\infty(\mathbb{D}^n)$, then*

$$\ker T_\varphi^* = \{0\}.$$

Proof. Note that $\ker T_\varphi^* = (\mathrm{ran} T_\varphi)^\perp$. But

$$\begin{aligned}\overline{\mathrm{ran} T_\varphi}^{H^2(\mathbb{D}^n)} &= \overline{I_{H^\infty(\mathbb{D}^n)}(\varphi)}^{H^2(\mathbb{D}^n)} \\ &= H^2(\mathbb{D}^n).\end{aligned}$$

The result now follows immediately. □

In the first part of the proof of Theorem 5.5, we verified that an M-ideal in $H^\infty(\mathbb{D}^n)$ is an invariant subspace of $H^2(\mathbb{D}^n)$. In this context, a subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is considered *invariant* if

$$z_i \mathcal{S} \subseteq \mathcal{S} \qquad (i = 1, \dots, n).$$

Corollary 5.7. *Let J be a nontrivial closed ideal in $H^\infty(\mathbb{D}^n)$. If there exists a proper closed invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ such that*

$$J \subseteq \mathcal{S},$$

then J is not an M-ideal in $H^\infty(\mathbb{D}^n)$.

Proof. Since $\mathcal{S}$ is a proper closed subspace of $H^2(\mathbb{D}^n)$, by Theorem 5.5, it follows that J is not an M-ideal; otherwise

$$H^2(\mathbb{D}^n) \supsetneqq \mathcal{S} \supseteq \overline{J}^{H^2(\mathbb{D}^n)} = H^2(\mathbb{D}^n),$$

a contradiction. □

Given a function $f \in H^2(\mathbb{D}^n)$, define

$$I_{H^2(\mathbb{D}^n)}(f) = \overline{\mathrm{span}\{pf : p \in \mathbb{C}[z_1, \dots, z_n]\}}^{H^2(\mathbb{D}^n)}.$$

In other words, $I_{H^2(\mathbb{D}^n)}(f)$ is the smallest closed invariant subspace of $H^2(\mathbb{D}^n)$ containing f.

Corollary 5.8. *Let $f \in H^\infty(\mathbb{D}^n)$. If $I_{H^\infty(\mathbb{D}^n)}(f)$ is an M-ideal in $H^\infty(\mathbb{D}^n)$, then*

$$I_{H^2(\mathbb{D}^n)}(f) = H^2(\mathbb{D}^n).$$

Proof. First, we observe that

$$I_{H^2(\mathbb{D}^n)}(f) = \overline{I_{H^\infty(\mathbb{D}^n)}(f)}^{H^2(\mathbb{D}^n)}.$$

Since $I_{H^\infty(\mathbb{D}^n)}(f)$ is an M-ideal in $H^\infty(\mathbb{D}^n)$, the result directly follows from Theorem 5.5. □

If $n = 1$, then a function $f \in H^\infty(\mathbb{D})$ is referred to as an *outer function* if

$$I_{H^2(\mathbb{D})}(f) = H^2(\mathbb{D}). \tag{5.3}$$

Consequently, Corollary 5.8 concludes that if $I_{H^\infty(\mathbb{D})}(f)$ is an M-ideal, then f is an outer function.

Note that the outer functions on the disc are zero-free. The notion of outer functions in higher variables is, however, different from the above (see Rudin [41, page 72]).

6. Principal M-ideals

From now on, we shall limit ourselves to single variable as a result of the unavailability of tools in several variables that are to be utilised in the subsequent computations. We begin with the final result, Corollary 5.8, of the previous section. It raises the question of a converse direction: Does the ideal generated by an outer function in $H^\infty(\mathbb{D})$ always qualify as an M-ideal in $H^\infty(\mathbb{D})$?

The main goal of this section is to prove that for functions in $Z^\infty(\mathbb{D})$, the converse is true. Recall from Definition 1.10 that $Z^\infty(\mathbb{D})$ is the set of all functions f in $H^\infty(\mathbb{D})$ that have a continuous extension to $Z_\mathbb{T}(f)$, where

$$Z_\mathbb{T}(f) = \{z \in \mathbb{T} : z \in Z_\mathbb{T}(f_O)\} \qquad (f \in H^\infty(\mathbb{D})),$$

and f_O denotes the outer factor of the inner-outer factorization of $f \in H^\infty(\mathbb{D})$. Note that

$$\tilde{f}(z) = \lim_{r \to 1^-} f(rz) \qquad (z \in \mathbb{T} \ a.e.),$$

is the radial limit extension of $f \in H^\infty(\mathbb{D})$ to the boundary $\mathbb{T}$. In view of the above definition of $Z^\infty(\mathbb{D})$, there may exist outer functions $f \in H^\infty(\mathbb{D})$ and points $z \in Z_\mathbb{T}(f)$ such that both $f(z) \neq 0$ and $\tilde{f}(z) \neq 0$. The point of considering functions in $Z^\infty(\mathbb{D})$ is that this does not happen here: let $f \in Z^\infty(\mathbb{D})$ and let $z \in Z_\mathbb{T}(f)$. Then, for any sequence $\{z_n\}$ converging to z, we have $f(z_n) \to 0$. In particular, the radial limit function satisfies the following:

$$\tilde{f}(z) = \lim_{r \to 1^-} f(rz) = 0.$$

If in addition, $f \in Z^\infty(\mathbb{D})$ is an outer function, then $f(z) = \tilde{f}(z) = 0$.

We recall representations of outer functions, which will play a crucial role in our analysis. For each $z \in \mathbb{D}$ and $\theta \in [-\pi, \pi]$, we define (in this context, also recall the Szegö kernel on $\mathbb{D}$)

$$S(z, \theta) = \frac{e^{i\theta} + z}{e^{i\theta} - z}.$$

It is well known that a function $f \in H^\infty(\mathbb{D})$ (or $f \in H^2(\mathbb{D})$) is outer if and only if (see [25, page 62]) there exists a real-valued integrable function k on $[-\pi, \pi]$ such that

$$f(z) = exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\theta)k(\theta)d\theta\right) \qquad (z \in \mathbb{D}). \tag{6.1}$$

Moreover, in this case

$$k(\theta) = \log|f(e^{i\theta})|,$$

for all $\theta \in [-\pi, \pi]$ a.e. In what follows, we will assume $Z_\mathbb{T}(f) \neq \emptyset$, otherwise f is invertible and

$$I_{H^\infty(\mathbb{D})}(f) = H^\infty(\mathbb{D}).$$

We need the following lemma:

Lemma 6.1. *Let $f \in Z^\infty(\mathbb{D})$ be an outer function. Then there exists a sequence of open sets $\{A_m\}_{m \geq 1}$ in $\mathbb{T}$ such that the following conditions hold:*

1. *$Z_\mathbb{T}(f) \subseteq A_m$ for all $m \geq 1$.*
2. *There exists an open set $U_m \subseteq \mathbb{T}$ such that $\overline{A_{m+1}} \subseteq U_m \subseteq A_m$ for all $m \geq 1$.*
3. *$\mu(A_m) \searrow 0$ as $m \to \infty$.*
4. *$|\tilde{f}| \leq e^{-m}$ on A_m for all $m \geq 1$.*

Proof. Consider the representation of the outer function f as in (6.1):

$$f(z) = exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\theta)k(\theta)d\theta\right) \qquad (z \in \mathbb{D}).$$

By the outer regularity of the Lebesgue measure, there exists a sequence of open sets $\{C_m\}$ in $\mathbb{T}$ such that

$$Z_{\mathbb{T}}(f) \subseteq C_m,$$

and

$$C_{m+1} \subseteq C_m,$$

and

$$\mu(C_m) \to \mu(Z_{\mathbb{T}}(f)) = 0.$$

For all $m \geq 1$. Choose a sequence of positive real numbers $\{\varepsilon_m\}_{m\geq 1}$ such that

$$\varepsilon_m < e^{-m} \qquad (m \geq 1).$$

Since f is continuously extendable to $Z_{\mathbb{T}}(f)$, corresponding to $\varepsilon_1 > 0$ and $\alpha \in Z_{\mathbb{T}}(f)$, there exists a $\delta_\alpha^1 > 0$ such that

$$|f(z) - f(\alpha)| < \varepsilon_1,$$

for all $z \in \mathbb{D} \cap B(\alpha, \delta_\alpha^1)$, and

$$\overline{B(\alpha_j, \delta_{\alpha_j}^1)} \cap \mathbb{T} \subsetneqq C_1,$$

where, for each $z_0 \in \mathbb{C}$ and $\delta > 0$, we write

$$B(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$

Since $f(\alpha) = 0$, it follows that

$$|f(z)| < \varepsilon_1,$$

for all $z \in \mathbb{D} \cap B(\alpha, \delta_\alpha^1)$. As $Z_{\mathbb{T}}(f)$ is a compact set, there exist scalars $\{\alpha_j\}_{j=1}^{n_1}$ such that

$$\{\alpha_j\}_{j=1}^{n_1} \subseteq Z_{\mathbb{T}}(f) \subseteq \bigcup_{j=1}^{n_1} B(\alpha_j, \delta_{\alpha_j}^1).$$

Define

$$B_{\varepsilon_1} = \bigcup_{j=1}^{n_1} B(\alpha_j, \delta_{\alpha_j}^1),$$

and

$$A_1 = \mathbb{T} \cap B_{\varepsilon_1}.$$

Now for $\varepsilon_2 > 0$ and for each $\alpha \in Z_{\mathbb{T}}(f)$, there exists a $\delta_\alpha^2 > 0$ such that

$$|f(z)| < \varepsilon_2,$$

for all $z \in \mathbb{D} \cap B(\alpha, \delta_\alpha^2)$, and

$$\overline{B(\alpha, \delta_\alpha^2)} \subsetneqq B_{\varepsilon_1}, \overline{B(\alpha, \delta_\alpha^2)} \cap \mathbb{T} \subsetneqq A_1 \cap C_2 \tag{6.2}$$

Again, by compactness of $Z_{\mathbb{T}}(f)$, there exist scalars $\{\alpha_j\}_{j=1}^{n_2}$ such that

$$\{\alpha_j\}_{j=1}^{n_2} \subseteq Z_{\mathbb{T}}(f) \subseteq \bigcup_{j=1}^{n_2} B(\alpha_j, \delta_{\alpha_j}^2).$$

Similarly, define

$$B_{\varepsilon_2} = \bigcup_{j=1}^{n_2} B(\alpha_j, \delta^2_{\alpha_j}),$$

and

$$A_2 = \mathbb{T} \cap B_{\varepsilon_2}.$$

Clearly

$$\overline{B}_{\varepsilon_2} \cap \mathbb{T} = \bigcup_{j=1}^{n_2} \left(\overline{B(\alpha_j, \delta^2_{\alpha_j})} \cap \mathbb{T}\right) \subsetneqq B_{\varepsilon_1} \cap \mathbb{T},$$

where the strict inclusion follows from the inclusions pointed out in (6.2) and the fact that B_{ε_1} is open. Therefore

$$\overline{A}_2 = \overline{B}_{\varepsilon_2} \cap \mathbb{T} \subsetneqq B_{\varepsilon_1} \cap \mathbb{T} = A_1.$$

Evidently, this process continues for $m > 2$. So we get a sequence $\{A_m\}_{m\geq 1}$ of open sets such that:
(1) $Z_{\mathbb{T}}(f) \subseteq A_m$, $m \geq 1$, and
(2) $\overline{A_{m+1}} \subseteq U_m \subseteq A_m$ for all $m \geq 1$,
which yields conditions (1) and (2). Also

$$0 \leq \mu(A_m) \leq \mu(C_m) \to 0,$$

implies (3). We now prove the remaining one, condition (4). Let $m \in \mathbb{N}$ be fixed. Suppose $\tilde{f}(e^{it})$ exists for some $e^{it} \in A_m$ (that is, $\tilde{f}(e^{it}) \in \mathbb{C}$). Then there exists $\alpha \in Z_{\mathbb{T}}(f)$ such that

$$e^{it} \in B(\alpha, \delta^m_\alpha).$$

Since $B(\alpha, \delta^m_\alpha)$ is open, there exists $0 < s < 1$ such that

$$re^{it} \in B(\alpha, \delta^m_\alpha) \cap \mathbb{D},$$

for all $s < r < 1$. Therefore

$$|f(re^{it})| < \varepsilon_m,$$

for all $s \leq r < 1$, and hence, the radial limit extension function $\tilde{f}$ satisfies the following property:

$$|\tilde{f}(e^{it})| < \varepsilon_m.$$

Since e^{it} is arbitrary, we get

$$\chi_{A_m} |\tilde{f}| < \varepsilon_m < e^{-m},$$

which yields (4), and completes the proof of the lemma. □

Now we are ready to prove the main result of this section. Recall that for $f \in H^\infty(\mathbb{D})$, we denote by $I_{H^\infty(\mathbb{D})}(f)$ the closed ideal generated by f.

Theorem 6.2. *Let $f \in Z^\infty(\mathbb{D})$. Then $I_{H^\infty(\mathbb{D})}(f)$ is an M-ideal in $H^\infty(\mathbb{D})$ if and only if f is an outer function.*

Proof. The necessary part is already in Corollary 5.8 (along with the identity (5.3)). For the sufficient part, assume that $f \in Z^\infty(\mathbb{D})$ is an outer function. Set

$$k(\theta) := \log |f(e^{i\theta})|,$$

for $\theta \in [-\pi, \pi]$ a.e. Then k is a real-valued integrable function, and

$$f(z) = exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\theta)k(\theta)d\theta\right) \qquad (z \in \mathbb{D}),$$

where $S(z,\theta) = \frac{e^{i\theta}+z}{e^{i\theta}-z}$ (see the discussion at the very start of this section). It is enough to build a bounded approximate unit in $I_{H^\infty(\mathbb{D})}(f)$ (see Theorem 2.3). We proceed as follows: By Lemma 6.1, we construct a sequence of open sets $\{A_m\}_{m\geq 1}$ in $\mathbb{T}$ such that

(1) $Z_{\mathbb{T}}(f) \subseteq A_m$ for all $m \geq 1$.
(2) $\overline{A_{m+1}} \subseteq U_m \subseteq A_m$, U_m open for all $m \geq 1$.
(3) $\mu(A_m) \searrow 0$ as $m \to \infty$.
(4) $|\tilde{f}| \leq e^{-m}$, or equivalently

$$k(\theta) < -m,$$

a.e. on A_m for all $m \geq 1$.

For each $m \geq 1$, define a real-valued integrable function k_m on $[-\pi, \pi]$ as

$$k_m = -\chi_{A_m^c} k.$$

For each $m \geq 1$, we claim that k_m is bounded above a.e. We prove this by contradiction. Fix $m \geq 1$. Assume, for each $N \in \mathbb{N}$, we have that

$$\mu\left(\{\theta : k_m(\theta) > N\} \cap A_m^c\right) > 0.$$

But

$$\begin{aligned}\mu\left(|\tilde{f}|^{-1}(0, e^{-N}) \cap A_m^c\right) &= \mu\left(\{\theta : k(\theta) < -N\} \cap A_m^c\right)\\ &= \mu\left(\{\theta : k_m(\theta) > N\} \cap A_m^c\right)\\ &> 0.\end{aligned}$$

Because of the inner regularity of Lebesgue measure, there exists a compact set B_N such that

$$B_N \subseteq |\tilde{f}|^{-1}(0, e^{-N}) \cap A_m^c,$$

and $\mu(B_N) > 0$. Since B_N is compact for each N, we can find $z_N \in B_N$ and a neighbourhood N_{z_N} of z_N such that $\mu(N_{z_N}) = e^{-N}$ and

$$\mu(N_{z_N} \cap B_N) > 0.$$

To see this, one may consider a cover by a set of open intervals of radius e^{-N} of B_N. Then, because B_N is compact, there exists a finite subcover. Since

$$\mu(B_N) > 0,$$

there exists at least one interval, say N_{z_N}, in the finite sub cover that intersects B_N with a positive measure. Take z_N to be the centre of this interval N_{z_N}. Passing onto a subsequence if necessary we can assume that $\{z_N\}$ converges to z. Let $\varepsilon > 0$ and let N_z be a neighbourhood of z. Without any loss of generality, we assume that N_z is centred at z. Then

$$|z_N - z| < \mu(N_z)/4,$$

for all $N > N_0$. Take N sufficiently large so that $N > N_0$ and

$$\mu(N_{z_N}) < \mu(N_z)/4,$$

and $e^{-N} < \varepsilon$. Then, for each $w \in N_{z_N}$, we have

$$\begin{aligned}|w-z| &\leq |w-z_N| + |z_N - z| \\ &< \mu(N_{z_n})/4 + \mu(N_z)/4 \\ &< \mu(N_z)/2.\end{aligned}$$

Therefore $N_{z_N} \subseteq N_z$, and hence $N_{z_N} \cap B_N \subseteq N_z$. Since $|\tilde{f}| < e^{-N} < \varepsilon$ a.e on B_N, it follows that

$$\mu(\{|\tilde{f}|^{-1}(0,\varepsilon) \cap N_z\}) \geq \mu(N_{z_N} \cap B_{N_0}) > 0.$$

This implies that A_m^c has a zero of f, which is impossible because $Z_{\mathbb{T}}(f) \subset A_m$. This proves the claim, and subsequently, we define outer function g_m by (recall the representations of outer functions from (6.1))

$$g_m(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\theta)k_m(\theta)d\theta\right) \qquad (z \in \mathbb{D}).$$

At this point, we recall that the real part of $S(z,\theta)$ is the Poisson kernel $P(z,\theta)$ (see [26, page 30] for more details). If we write $z = re^{i\xi}$, then as $r \to 1$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\operatorname{Re} S(re^{i\xi},\theta)\Big]k_m(\theta)\,d\theta \to k_m(\xi),$$

a.e. Now

$$\begin{aligned}|\tilde{g}_m(e^{i\xi})| &= \left|\lim_{r\to 1}\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(re^{i\xi},\theta)k_m(\theta)d\theta\right)\right| \\ &= \lim_{r\to 1}\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\xi},\theta)k_m(\theta)d\theta\right) \\ &= \exp\left(\frac{1}{2\pi}\lim_{r\to 1}\int_{-\pi}^{\pi} P(re^{i\xi},\theta)(-\chi_{A_m^c}(e^{i\theta})k(\theta))d\theta\right) \\ &= \exp\left(-\chi_{A_m^c}(e^{i\xi})k(\xi)\right) \\ &= \exp\Big(k_m(\xi)\Big),\end{aligned}$$

a.e. Since k_m is bounded, it follows that $g_m \in H^\infty(\mathbb{D})$. Again, we compute

$$\begin{aligned}(fg_m)(re^{i\xi}) &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(re^{i\xi},\theta)k(\theta)d\theta\right) \times \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(re^{i\xi},\theta)k_m(\theta)d\theta\right) \\ &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(re^{i\xi},\theta)k(\theta)d\theta\right) \times \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(re^{i\xi},\theta) - \chi_{A_m^c}(e^{i\theta})k(\theta)d\theta\right) \\ &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(re^{i\xi},\theta)\left(k(\theta) - \chi_{A_m^c}(e^{i\theta})k(\theta)\right)d\theta\right) \\ &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(re^{i\xi},\theta)\chi_{A_m}(e^{i\theta})k(\theta)d\theta\right).\end{aligned}$$

We now claim that $\{fg_m\}_{m\geq 1}$ is a bounded approximate unit in $I_{H^\infty(\mathbb{D})}(f)$. For fixed $m \geq 1$ and $e^{i\xi} \in \mathbb{T}$, we have

$$\begin{aligned}
|\widetilde{(fg_m)}(e^{i\xi})| &= \left| \lim_{r\to 1} \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} S(re^{i\xi},\theta)\chi_{A_m}(e^{i\theta})k(\theta)d\theta \right) \right| \\
&= \lim_{r\to 1} \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} P(re^{i\xi},\theta)\chi_{A_m}(e^{i\theta})k(\theta)d\theta \right) \\
&= \exp\left(\frac{1}{2\pi} \lim_{r\to 1} \int_{-\pi}^{\pi} P(re^{i\xi},\theta)\chi_{A_m}(e^{i\theta})k(\theta)d\theta \right) \\
&= \exp\left(\chi_{A_m}(e^{i\xi})k(\xi) \right),
\end{aligned}$$

a.e. But, on one hand

$$\chi_{A_m}(e^{i\xi})k(\xi) = 0,$$

for all $e^{i\xi} \in A_m^c$ a.e. and, on the other hand, by property (4) above, we know

$$\chi_{A_m}(e^{i\xi})k(\xi) \leq -m,$$

for all $e^{i\xi} \in A_m$ a.e. Therefore

$$\begin{cases} |\widetilde{fg}_m(e^{i\xi})| = 1 & \text{if } e^{i\xi} \in A_m^c \\ |\widetilde{fg}_m(e^{i\xi})| \leq e^{-m} & \text{if } e^{i\xi} \in A_m. \end{cases} \tag{6.3}$$

We want to show that $fg_m \to 1$ uniformly on $A_{m_0}^c$ for each $m_0 \geq 1$. To see this, fix $m_0 \geq 1$. We know that $A_m \subsetneqq A_{m_0}$ for all $m > m_0$. Also, by property (2), there exists $M > 0$ such that

$$|e^{i\theta} - e^{i\xi}| > M,$$

for all $e^{i\theta} \in A_m$ and $e^{i\xi} \in A_{m_0}^c$. Therefore, there exists $\tilde{M} > 0$ such that

$$|S(re^{i\xi},\theta)\chi_{A_m}(e^{i\theta})| \leq \tilde{M},$$

for all $e^{i\xi} \in A_{m_0}^c$ a.e. and $m > m_0$. Since

$$\chi_{A_m}(e^{i\theta})|k(\theta)| \longrightarrow 0,$$

pointwise, and

$$\chi_{A_m}(e^{i\theta})|k(\theta)| \leq |k(\theta)|,$$

on $\mathbb{T}$ a.e. by the dominated convergence theorem, we conclude

$$\int_{-\pi}^{\pi} \chi_{A_m}(e^{i\theta})|k(\theta)|d\theta \longrightarrow 0.$$

Consequently, for each $m > m_0$, we have

$$\left| \int_{-\pi}^{\pi} S(re^{i\xi},\theta)\chi_{A_m}(e^{i\theta})k(\theta)d\theta \right| \leq \tilde{M} \int_{-\pi}^{\pi} \chi_{A_m}(e^{i\theta})|k(\theta)|d\theta \longrightarrow 0,$$

uniformly for all $e^{i\xi} \in A_{m_0}^c$ a.e. Since

$$\widetilde{(fg_m)}(e^{i\xi}) = \lim_{r\to 1} \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} S(re^{i\xi},\theta)\chi_{A_m}(e^{i\theta})k(\theta)d\theta \right),$$

we conclude that

$$\widetilde{fg_m} \longrightarrow 1,$$

uniformly on $A^c_{m_0}$ a.e. and hence, the claim follows. Finally, we turn to compute $\|f^2g_m - f\|$. Observe that

$$\|f^2g_m - f\| = \max\Big\{\operatorname*{ess\,sup}_{A^C_{m_0}} |\widetilde{f^2g_m} - \tilde{f}|, \operatorname*{ess\,sup}_{A_m} |\widetilde{f^2g_m} - \tilde{f}|\Big\}.$$

By the fact of uniform convergency above, for sufficiently large m, it follows that

$$\operatorname*{ess\,sup}_{A^C_{m_0}} |\widetilde{f^2g_m} - \tilde{f}| \leq \varepsilon.$$

Therefore, for sufficiently large m, property (4) and (6.3) imply

$$\operatorname*{ess\,sup}_{A_m} |\widetilde{f^2g_m} - \tilde{f}| \leq e^{-m}(e^{-m} + 1),$$

and hence $\{fg_m\}_{m\geq 1}$ is a bounded approximate unit in $I_{H^\infty(\mathbb{D})}(f)$. $\square$

We conclude this section by computing the zero set of a specific example. This will be utilized for additional computations in the following section. It is convenient to recall the representations of outer functions from (6.1): Define an integrable function k on $(-\pi, \pi]$ as follows:

$$k(\theta) = \begin{cases} -n & \text{if } \theta \in (\frac{1}{(n+1)}, \frac{1}{n}] \cup [\frac{-1}{n}, \frac{-1}{(n+1)}) \\ \frac{1}{n} & \text{if } \theta \in [\pi - \frac{1}{2^n}, \pi - \frac{1}{2^n} - \frac{1}{8^n}] \\ 1 & \text{otherwise .} \end{cases}$$

Define the outer function corresponding to k as

$$f(z) = exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\theta)k(\theta)d\theta\right) \qquad (z \in \mathbb{D}), \tag{6.4}$$

where $S(z,\theta) = \frac{e^{i\theta}+z}{e^{i\theta}-z}$. We claim that

$$Z_{\mathbb{T}}(f) = \{1\}.$$

Indeed, since

$$|\tilde{f}(e^{i\theta})| = e^{k(\theta)},$$

for $\theta \in (-\pi, \pi]$ a.e. it follows that

$$|\tilde{f}(e^{i\theta})| \geq 1,$$

for all $\theta \notin (-1/2, 1/2))$. Clearly, for any $z = e^{i\theta}$ such that $\theta \notin (-1/2, 1/2))$, and each neighbourhood N_z of z, we have

$$|\tilde{f}|^{-1}(0, 1/2) \cap N_z = \emptyset.$$

Therefore

$$e^{i\theta} \notin Z_{\mathbb{T}}(f),$$

for all $\theta \notin (-1/2, 1/2))$. Now we consider the case

$$\theta \in (-1/2, 1/2)) \setminus \{0\}.$$

There exists an $n \geq 1$ such that

$$\theta \in \Big(\frac{1}{(n+1)}, \frac{1}{n}\Big] \bigcup \Big[\frac{-1}{n}, \frac{-1}{(n+1)}\Big).$$

Hence

$$|\tilde{f}(e^{i\zeta})| = e^{-n},$$

for all $\zeta \in (\frac{1}{(n+1)}, \frac{1}{n}] \cup [\frac{-1}{n}, \frac{-1}{(n+1)})$. For $z = e^{i\theta}$, and $N_z = (\frac{1}{(n+1)}, \frac{1}{n}] \cup [\frac{-1}{n}, \frac{-1}{(n+1)})$, and $\varepsilon < e^{-n}$, we get

$$|\tilde{f}|^{-1}(0, \varepsilon) \cap N_z = \emptyset.$$

Therefore, we again conclude that $e^{i\theta} \notin Z_{\mathbb{T}}(f)$ for all $\theta \in (-1/2, 1/2)) \setminus \{0\}$, and hence

$$e^{i\theta} \notin Z_{\mathbb{T}}(f) \qquad (\theta \in [-\pi, \pi) \setminus \{0\}).$$

Finally, for each $\varepsilon > 0$ and neighbourhood N_1 of 1, we can find $n \geq 1$ such that $e^{-n} < \varepsilon$ and $(\frac{1}{(n+1)}, \frac{1}{n}] \cup [\frac{-1}{n}, \frac{-1}{(n+1)}) \subseteq N_1$. Therefore,

$$\Big(\frac{1}{(n+1)}, \frac{1}{n}\Big] \bigcup \Big[\frac{-1}{n}, \frac{-1}{(n+1)}\Big) \subseteq |\tilde{f}|^{-1}(0, \varepsilon) \cap N_1,$$

and hence

$$\mu(|\tilde{f}|^{-1}(0, \varepsilon) \cap N_1) > 0.$$

This completes the proof of the claim that $Z_{\mathbb{T}}(f) = \{1\}$.

7. Examples

The objective of this section is to present some direct applications of Theorem 6.2 as well as some nontrivial examples of elements in $Z^\infty(\mathbb{D})$. Additionally, we present exotic examples of M-ideals that are singly generated by functions from $H^\infty(\mathbb{D}) \setminus Z^\infty(\mathbb{D})$. We begin by highlighting, in the context of Theorem 6.2, that the M-ideals are more explicit. For instance, given $f \in Z^\infty(\mathbb{D})$, define

$$\mathcal{I}(f) = \{g \in H^\infty(\mathbb{D}) : Z_{\mathbb{T}}(f) \subseteq Z_{\mathbb{T}}(g) \text{ and } g \text{ has a continuous extension to } Z_{\mathbb{T}}(f)\}. \tag{7.1}$$

Observe that

$$I_{H^\infty(\mathbb{D})}(f) \subseteq \mathcal{I}(f).$$

Now we assume that f is outer. Pick $g \in H^\infty(\mathbb{D})$ such that $\tilde{g}|_{Z_{\mathbb{T}}(f)}$ is continuous, and

$$Z_{\mathbb{T}}(f) \subseteq Z_{\mathbb{T}}(g).$$

For each open set U containing $Z_{\mathbb{T}}(f)$, there exists a neighborhood $A \subseteq U$ of $Z_{\mathbb{T}}(f)$ such that the bounded approximate unit $\{fg_m\}_{m \geq 1}$ constructed in the proof of Theorem 6.2 converges uniformly to 1 on A^c and

$$|\tilde{f}|, |\tilde{g}| < \varepsilon,$$

a.e. on A. Since

$$\widetilde{fg_m} \longrightarrow 1,$$

uniformly on A^c, it follows that

$$\operatorname*{ess\,sup}_{e^{i\theta} \in A^c} |\tilde{g}(e^{i\theta})(\widetilde{fg_m})(e^{i\theta}) - \tilde{g}(e^{i\theta})| < \varepsilon,$$

and

$$\operatorname*{ess\,sup}_{e^{i\theta} \in A} |\tilde{g}(e^{i\theta})(\widetilde{fg_m})(e^{i\theta}) - \tilde{g}(e^{i\theta})| < |\tilde{g}(\theta)| \|fg_m + 1\| < 2\varepsilon,$$

for sufficiently large m. Consequently, $g \in I_{H^\infty(\mathbb{D})}(f)$. This proves the following result:

Corollary 7.1. *If $f \in Z^\infty(\mathbb{D})$ is an outer function, then*

$$I_{H^\infty(\mathbb{D})}(f) = \mathcal{I}(f).$$

In view of $\mathbb{C}[z] \subseteq Z^\infty(\mathbb{D})$, the following is immediate from Theorem 6.2:

Corollary 7.2. *Let $p \in \mathbb{C}[z]$ be a polynomial. Then $I_{H^\infty(\mathbb{D})}(p)$ is an M-ideal in $H^\infty(\mathbb{D})$ if and only if zeros of p lies on $\mathbb{C} \setminus \mathbb{D}$.*

We now go on to the existence of some nontrivial outer functions in $Z^\infty(\mathbb{D})$. We provide two classes of examples of different flavors. Recall that the disc algebra $A(\mathbb{D})$ is the uniform algebra of all continuous functions on the closed disc $\overline{\mathbb{D}}$ that are analytic on the open disc $\mathbb{D}$. In other words (see (1.4))

$$A(\mathbb{D}) = H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}}).$$

It is known, as well as evident, that $A(\mathbb{D}) \subsetneqq H^\infty(\mathbb{D})$. In the following, we present an example of an outer function in $Z^\infty(\mathbb{D})$ that also has points of discontinuity on $\mathbb{T}$. In particular, we prove that $A(\mathbb{D}) \subsetneqq Z^\infty(\mathbb{D})$.

Example 7.3. *Consider the singular inner function*

$$s(z) = \exp\left(\frac{z+i}{z-i}\right) \qquad (z \in \mathbb{D}),$$

and the outer function

$$h(z) = 1 - z \qquad (z \in \mathbb{D}).$$

Since s is an inner function, $1 - s$ is an outer function. Therefore, being the product of two outer functions, it follows that

$$f(z) = (1 - s(z))h(z) \qquad (z \in \mathbb{D}),$$

is an outer function. Observe that

$$f(z) = (1-z)\Big(1 - \exp\left(\frac{z+i}{z-i}\right)\Big) \qquad (z \in \mathbb{D}),$$

and hence

$$Z_{\mathbb{T}}(f) = \{1, -i\}.$$

Since the support of $\tilde{s}$ is $\{i\}$, it follows that s is analytically extendable to $\mathbb{T}\backslash\{i\}$. In particular, $1 - s$ is continuously extendable to $\{1, -i\}$. Since s is continuous on $\mathbb{T}\backslash\{i\}$, it follows that f is continuous on $Z_{\mathbb{T}}(f)$. Therefore $f \in Z^\infty(\mathbb{D})$. However, since s is discontinuous at i and h does not vanish at the point i, we conclude that f is discontinuous at i, and subsequently, $f \notin A(\mathbb{D})$.

Now we turn to the second example.

Example 7.4. *We consider the outer function f as constructed in* (6.4)*:*

$$f(z) = exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\theta)k(\theta)d\theta\right) \qquad (z \in \mathbb{D}),$$

where

$$k(\theta) = \begin{cases} -n & \text{if } \theta \in (\frac{1}{(n+1)}, \frac{1}{n}] \cup [\frac{-1}{n}, \frac{-1}{(n+1)}) \\ \frac{1}{n} & \text{if } \theta \in [\pi - \frac{1}{2^n}, \pi - \frac{1}{2^n} - \frac{1}{8^n}] \\ 1 & \text{otherwise} . \end{cases}$$

We already know that $Z_{\mathbb{T}}(f) = \{1\}$. Observe, for sufficiently large n, we have

$$\theta \in \left[\frac{-1}{n}, \frac{-1}{(n+1)}\right) \cup \left(\frac{1}{(n+1)}, \frac{1}{n}\right],$$

whenever θ gets close to 0. Hence

$$|\tilde{f}(e^{i\theta})| = e^{k(\theta)} = e^{-n}.$$

As a result, $\tilde{f}$ is continuous at 1. Now there exist sequences $\{\theta_m\}$ and $\{\gamma_m\}$ such that

$$\theta_m \subseteq \left[\pi - \frac{1}{2^m}, \pi - \frac{1}{2^m} - \frac{1}{8^m}\right] \qquad (m \geq 1),$$

and

$$k(\gamma_m) = 1 \qquad (m \geq 1),$$

and $\theta_m \to \pi$, and $\gamma_m \to \pi$. This implies that $|\tilde{f}|$ is not continuous at π.

Therefore, there is no dearth of examples of functions in $Z^\infty(\mathbb{D})$. Moreover, as we have shown above

$$A(\mathbb{D}) \subsetneqq Z^\infty(\mathbb{D}).$$

We now present an additional collection of natural yet exotic M-ideals in $H^\infty(\mathbb{D})$.

Theorem 7.5. *Let $f \in H^\infty(\mathbb{D})$ such that $\|f\| = 1$. Suppose $|\alpha| = 1$ and*

$$\alpha \in \text{ess-ran}\, f.$$

Then $I_{H^\infty(\mathbb{D})}(\alpha - f)$ is an M-ideal in $H^\infty(\mathbb{D})$.

Proof. Without any loss of generality, we may assume that

$$\alpha = 1 \in \text{ess-range} f.$$

Fix an $\varepsilon > 0$, and let

$$A = \{\zeta \in \mathbb{T} : \tilde{f}(\zeta) \in B_\varepsilon(1)\},$$

where $B_\varepsilon(1)$ denotes the open ball of radius ε centered at 1. Note that

$$\tilde{f}(A^c) \subseteq \overline{\mathbb{D}} \setminus B_\varepsilon(1).$$

Now we define a function $g \in H^\infty(\mathbb{D})$ by

$$g = \frac{1+f}{2}.$$

For each $z \in A$, we have

$$|1 - g(z)| = \left|1 - \frac{1+f(z)}{2}\right| = \frac{1}{2}\,|1 - f(z)| < \frac{\varepsilon}{2},$$

and hence

$$\tilde{g}(A) \subseteq B_{\frac{\varepsilon}{2}}(1).$$

Define $h \in A(\mathbb{D})$ by

$$h := \frac{z+1}{2}.$$

We have by the triangle inequality that $\|h\| \leq 1$. Moreover, on the boundary $\mathbb{T}$, we have

$$|h(e^{i\theta})|^2 = \frac{2(1+cos(\theta))}{4} = \frac{1+cos(\theta)}{2}.$$

Therefore $|h(e^{i\theta})| = 1$ only when $e^{i\theta} = 1$. By continuity, there exists $c \in (0,1)$ such that

$$\sup_{z \in \overline{\mathbb{D}} \setminus B_\varepsilon(1)} |h(z)| \leq c.$$

Since
$$\tilde{g} = h \circ \tilde{f},$$
it follows that
$$|\tilde{g}| \leq c < 1,$$
on A^c. Observe that
$$I_{H^\infty(\mathbb{D})}(1-f) = I_{H^\infty(\mathbb{D})}(1-g).$$
Therefore, it is enough to prove that $I_{H^\infty(\mathbb{D})}(1-g)$ is an M-ideal in $H^\infty(\mathbb{D})$. To this end, we will construct one approximate identity in $I_{H^\infty(\mathbb{D})}(1-g)$. For each $n \geq 1$, set
$$f_n = 1 - g^n.$$
Since
$$1 - g^n = (1-g)(1 + g + \cdots + g^{n-1}),$$
it follows that $f_n \in I_{H^\infty(\mathbb{D})}(1-g)$. Moreover
$$(1-g)f_n - (1-g) = -(1-g)g^n.$$
By using the property of radial limits, we conclude
$$|(1-g)g^n| \leq \begin{cases} \frac{\varepsilon}{2} & \text{on } A \\ 2c^n & \text{on } A^c. \end{cases}$$
By the maximum modules principle, we find that $\{f_n\}_{n\geq 1}$ is a bounded approximate identity in $I_{H^\infty(\mathbb{D})}(1-g)$, and hence $I_{H^\infty(\mathbb{D})}(1-g)$ is an M-ideal in $H^\infty(\mathbb{D})$. $\square$

Now we turn to the final example in this section. The class of functions given in the above theorem has the potential for the existence of singly generated M-ideal in $H^\infty(\mathbb{D})$ that may not belong to $Z^\infty(\mathbb{D})$. Indeed, the following example shows an M-ideal in $H^\infty(\mathbb{D})$ generated by an outer function not belonging to the class $Z^\infty(\mathbb{D})$.

Example 7.6. *Define an integrable function k on $(-\pi, \pi]$ by*
$$k(\theta) = \begin{cases} -(\theta^2+1) & \text{if } -\pi < \theta < 0 \\ -\theta & \text{if } 0 \leq \theta \leq \pi, \end{cases}$$
and consider the corresponding outer function
$$f(z) = exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\theta)k(\theta)d\theta\right) \qquad (z \in \mathbb{D}).$$
Now, observe that
$$1 \in \textit{ess-range}|\tilde{f}|,$$
and hence there exists $\alpha \in \mathbb{T}$ such that
$$1 \in Z_{\mathbb{T}}(\alpha - f).$$
On the other hand, if $\theta \to 0$ from the left, then
$$k(\theta) \longrightarrow -1,$$
and hence
$$|\tilde{f}| \longrightarrow k(\theta) = \frac{1}{e}.$$

Therefore, $\alpha - f$ is not continuous at 1 *(or on $Z_{\mathbb{T}}(\alpha - f)$) but, in view of the above theorem, $I_{H^\infty(\mathbb{D})}(\alpha - f)$ remains an M-ideal in $H^\infty(\mathbb{D})$.*

The examples and results shown in this section indicate that the structure of M-ideals in $H^\infty(\mathbb{D})$ is intricate, even when considering M-ideals that are singly generated.

8. Finitely generated M-ideals

In the preceding sections, we examined representations of ideals that are singly generated by functions from $Z^\infty(\mathbb{D})$. Specifically, we showed in Theorem 6.2 that the ideal $I_{H^\infty(\mathbb{D})}(f)$ is an M-ideal for some function $f \in Z^\infty(\mathbb{D})$ if and only if f is an outer function. In this section, we consider M-ideals in $H^\infty(\mathbb{D})$ that are finitely generated by functions from $Z^\infty(\mathbb{D})$. Given m functions $\{f_1, \ldots, f_m\} \subseteq H^\infty(\mathbb{D})$, denote by $I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m)$ the ideal generated by $\{f_1, \ldots, f_m\}$.

Recall the inner-outer factorizations of $H^\infty(\mathbb{D})$-functions (see (1.3)): Given a nonzero function $f \in H^\infty(\mathbb{D})$, there exist an inner function f_I and an outer function f_O such that

$$f = f_I f_O,$$

on $\mathbb{D}$. Moreover, this factorization is unique up to a constant of modulus one.

Theorem 8.1. *Let $\{f_1, \ldots, f_m\} \subseteq Z^\infty(\mathbb{D})$. If $I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m)$ is an M-ideal in $H^\infty(\mathbb{D})$, then there exists an outer function $f \in Z^\infty(\mathbb{D})$ such that*

$$I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m) = I_{H^\infty(\mathbb{D})}(f).$$

Proof. Suppose $I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m)$ is an M-ideal in $H^\infty(\mathbb{D})$. Let

$$f_j = f_{Ij} f_{Oj},$$

is the inner-outer factorization, where $f_{I,j}$ is the inner factor and $f_{O,j}$ is the outer factor of f_j for all $j = 1, \ldots, m$. In view of

$$f_{Ij} f_{Oj} \in I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m) \qquad (j = 1, \ldots, m),$$

we know, by Theorem 5.2, that

$$f_{Oj} \in I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m) \qquad (j = 1, \ldots, m),$$

and hence

$$I_{H^\infty(\mathbb{D})}(f_{O1}, \ldots, f_{Om}) \subseteq I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m).$$

By using the inner-outer factorizations of f_i's, the other set inclusion becomes trivial. Then

$$I_{H^\infty(\mathbb{D})}(f_{O1}, \ldots, f_{Om}) = I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m).$$

Therefore, without any loss of generality, we may assume that f_j's are outer functions in $H^\infty(\mathbb{D})$. Our goal is to prove that

$$I_{H^\infty(\mathbb{D})}(f_1, \ldots, f_m) = I_{H^\infty(\mathbb{D})}(f),$$

for some outer function $f \in Z^\infty(\mathbb{D})$. We prove the assertion for $m = 2$. The complete proof follows similarly by induction on m. Therefore, our revised goal is to prove the following fact: Assuming that $I_{H^\infty(\mathbb{D})}(f_1, f_2)$ is an M-ideal in $H^\infty(\mathbb{D})$ for some $f_1, f_2 \in Z^\infty(\mathbb{D})$, there exists an outer function $f \in Z^\infty(\mathbb{D})$ such that

$$I_{H^\infty(\mathbb{D})}(f_1, f_2) = I_{H^\infty(\mathbb{D})}(f).$$

To this end, first, assume that

$$Z(f_1) \cap Z(f_2) = \emptyset.$$

Since $f_j \in Z^\infty(\mathbb{D})$ is outer, by Theorem 6.2, it follows that $I_{H^\infty(\mathbb{D})}(f_j)$ is an M-ideal, $j = 1, 2$, and hence there exist bounded approximate units, say (by an abuse of notation) $\{\varphi_m\}_{m\geq 1}$ and $\{\psi_m\}_{m\geq 1}$ in $I_{H^\infty(\mathbb{D})}(f_1)$ and $I_{H^\infty(\mathbb{D})}(f_2)$, respectively, as constructed in the proof of Theorem 6.2. We highlight the following key properties: For any $\varepsilon > 0$, there exists a neighborhood $U_1 \subseteq \mathbb{T}$ of $Z_{\mathbb{T}}(f_1)$ such that

$$|\widetilde{\varphi_m} - 1| < \varepsilon, \tag{8.1}$$

on U_1^c, and

$$|\widetilde{\varphi_m}| < \varepsilon, \tag{8.2}$$

on U_1 for sufficiently large m. Similarly, there exists a neighborhood $U_2 \subseteq \mathbb{T}$ of $Z_{\mathbb{T}}(f_2)$ such that $|\widetilde{\varphi_m} - 1| < \varepsilon$ on U_1^c, and $|\widetilde{\varphi_m}| < \varepsilon$ on U_2 for sufficiently large m. Since $Z_{\mathbb{T}}(f_1)$ and $Z_{\mathbb{T}}(f_2)$ are compact, without any loss of generality, we may assume that

$$U_1 \cap U_2 = \emptyset.$$

Therefore, for sufficiently large $p, q \geq 1$, we have

$$\Big|\frac{\widetilde{\varphi}_p + \widetilde{\psi}_q}{2} - 1\Big| < \varepsilon,$$

on $U_1^c \cup U_2^c = \mathbb{T}$. That is

$$1 - \varepsilon < \Big|\frac{\widetilde{\varphi}_p + \widetilde{\psi}_q}{2}\Big|,$$

on $\mathbb{T}$. But

$$\frac{\widetilde{\varphi}_p + \widetilde{\psi}_q}{2} \in I_{H^\infty(\mathbb{D})}(f_1, f_2),$$

and hence, by Theorem 5.2, we conclude

$$I_{H^\infty(\mathbb{D})}(f_1, f_2) = I_{H^\infty(\mathbb{D})}(1).$$

Now we assume that

$$Z_{\mathbb{T}}(f_1) \cap Z_{\mathbb{T}}(f_2) \neq \emptyset.$$

For each $p, q \geq 1$, define

$$\zeta_{p,q} = \varphi_p + \psi_q - \varphi_p\psi_q.$$

Then

$$\zeta_{p,q} \in I_{H^\infty(\mathbb{D})}(f_1, f_2) \qquad (p, q \geq 1).$$

We claim that $\{\zeta_{p,q}\}_{p,q\geq 1}$ is a bounded approximate unit in $I_{H^\infty(\mathbb{D})}(f_1, f_2)$. It is clear that $\{\zeta_{p,q}\}_{p,q\geq 1}$ is a bounded set. It is now enough to prove that

$$\lim_{p,q} \zeta_{p,q} f_j = f_j,$$

for all $j = 1, 2$ (with respect to the lexicographic ordering). Fix $\varepsilon > 0$. Since $\{\varphi_m\}_{m\geq 1}$ is a bounded approximate identity in $I_{H^\infty(\mathbb{D})}(f_1)$, there exists $p_0 \geq 1$ such that

$$\|f_1 - \varphi_p f_1\|_\infty \leq \varepsilon \qquad (p \geq p_0).$$

We compute

$$\begin{aligned}\|\zeta_{p,q}f_1 - f_1\| &= \|\varphi_p f_1 + \psi_q f_1 - \varphi_p\psi_q f_1 - f_1\|\\ &= \|\varphi_p f_1 - f_1 + \psi_q f_1 - \varphi_p\psi_q f_1\|\\ &\leq \|\varphi_p f_1 - f_1\| + \|\psi_q\|\|f_1 - \varphi_p f_1\|\\ &\leq \varepsilon(1+\|\psi_q\|)\\ &\leq \varepsilon(1+M),\end{aligned}$$

for all $p \geq p_0$, where (recall that $\{\psi_q\}_{q\geq 1}$ is a bounded sequence)

$$M = \sup_q \|\psi_q\|.$$

Assume, without any lose of generality, that $\|\varphi_p\|_\infty \leq M$ for all $p \geq 1$. Then, a similar computation as above implies the existence of a number $q_0 \geq 1$ such that

$$\|\zeta_{p,q}f_2 - f_2\| \leq \varepsilon(1+M),$$

for all $q \geq q_0$. Hence

$$\|\zeta_{p,q}f_j - f_j\| \leq \varepsilon(1+M),$$

for all $j = 1, 2$, and $p \geq p_0$ and $q \geq q_0$. Then $\{\zeta_{p,q}\}_{p,q\geq 1}$ is a bounded approximate identity in, and consequently $I_{H^\infty(\mathbb{D})}(f_1, f_2)$ is an M-ideal. It remains to prove that $I_{H^\infty(\mathbb{D})}(f_1, f_2) = I_{H^\infty(\mathbb{D})}(f)$ for some outer function $f \in H^\infty(\mathbb{D})$. First, let us define

$$\mathcal{I} = \{h \in H^\infty(\mathbb{D}) : Z_{\mathbb{T}}(f_1) \cap Z_{\mathbb{T}}(f_2) \subseteq Z_{\mathbb{T}}(h) \text{ and } h|_{Z_{\mathbb{T}}(f_1)\cap Z_{\mathbb{T}}(f_2)} \text{ is continuous}\}.$$

We claim that $I_{H^\infty(\mathbb{D})}(f_1, f_2) = \mathcal{I}$. Let $h \in \mathcal{I}$. We want to show that

$$\lim_{p,q} \widetilde{\zeta_{p,q}}\tilde{h} = \tilde{h}.$$

Let $\varepsilon > 0$. Since $h \in C(Z(f_1) \cap Z(f_2))$, we can find an open set $U \subset \mathbb{T}$ such that

$$Z(f_1) \cap Z(f_2) \subset U,$$

and

$$|\tilde{h}| < \frac{\varepsilon}{4},$$

on U. As in (8.1) and (8.2), there exist open sets $U_1, U_2 \subseteq \mathbb{T}$ such that $Z(f_j) \subseteq U_j$, $j = 1, 2$, and

$$U_1 \cap U_2 \subseteq U.$$

Moreover, we have the following properties:

$$\frac{\varepsilon}{2\|\tilde{h}\|} > \begin{cases} |\widetilde{\varphi_p} - 1| & \text{on } U_1^c \\ |\widetilde{\varphi_p}| & \text{on } U_1, \end{cases}$$

and

$$\frac{\varepsilon}{2\|\tilde{h}\|} > \begin{cases} |\widetilde{\psi_q} - 1| & \text{on } U_2^c \\ |\widetilde{\psi_q}| & \text{on } U_2, \end{cases}$$

for sufficiently large p and q. Also

$$\|\widetilde{\zeta_{p,q}}\tilde{h} - \tilde{h}\| = \|(\widetilde{\varphi_p}\tilde{h} + \widetilde{\psi_q}\tilde{h} - \widetilde{\varphi_p}\widetilde{\psi_q}\tilde{h}) - \tilde{h}\|,$$

where, for sufficiently large p and q, we have

$$
\begin{aligned}
\|(\widetilde{\varphi_p}\tilde{h} + \widetilde{\psi_q}\tilde{h} - \widetilde{\varphi_p}\widetilde{\psi_q}\tilde{h}) - \tilde{h}\| &\leq \begin{cases} 4\|\tilde{h}\| & \text{on } U_1 \cap U_2 \\ \|\widetilde{\varphi_p}\|\|\tilde{h} - \widetilde{\psi_q}\tilde{h}\| + \|\widetilde{\psi_q}\tilde{h} - \tilde{h}\| & \text{on } U_1 \cap U_2^c \\ \|\widetilde{\psi_q}\|\|\tilde{h} - \widetilde{\varphi_p}\tilde{h}\| + \|\widetilde{\varphi_p}\tilde{h} - \tilde{h}\| & \text{on } U_1^c \cap U_2 \\ \|\widetilde{\psi_q}\|\|\tilde{h} - \widetilde{\varphi_p}\tilde{h}\| + \|\widetilde{\varphi_p}\tilde{h} - \tilde{h}\| & \text{on } U_1^c \cap U_2^c \end{cases} \\
&\leq \begin{cases} \varepsilon & \text{on } U_1 \cap U_2 \\ \frac{\varepsilon}{2\|\tilde{h}\|} \cdot \|\tilde{h}\| + \frac{\varepsilon}{2\|\tilde{h}\|} \cdot \|\tilde{h}\| & \text{on } U_1 \cap U_2^c \\ \frac{\varepsilon}{2\|\tilde{h}\|} \cdot \|\tilde{h}\| + \frac{\varepsilon}{2\|\tilde{h}\|} \cdot \|\tilde{h}\| & \text{on } U_1^c \cap U_2 \\ \frac{\varepsilon}{2\|\tilde{h}\|} \cdot \|\tilde{h}\| + \frac{\varepsilon}{2\|\tilde{h}\|} \cdot \|\tilde{h}\| & \text{on } U_1^c \cap U_2^c \end{cases} \\
&< \varepsilon,
\end{aligned}
$$

and hence

$$h \in I_{H^\infty(\mathbb{D})}(f_1, f_2).$$

On the other hand, it is clear that every function in $I(f_1, f_2)$ vanishes and is continuous on $Z(f_1) \cap Z(f_2)$. Therefore

$$I_{H^\infty(\mathbb{D})}(f_1, f_2) = \mathcal{I},$$

which completes the proof of the claim. Finally, since f_1 and f_2 are continuous on $Z(f_1) \cap Z(f_2)$ and since $Z(f_1) \cap Z(f_2)$ is a closed subset of $\mathbb{T}$ with Lebesgue measure 0, by a theorem of Fatou, there exists an outer function $f \in A(\mathbb{D})$ such that $Z_f = Z(f_1) \cap Z(f_2)$. But Corollary 7.1 implies

$$\mathcal{I}(f) = I(f_1, f_2),$$

which completes the proof of the result. □

It is perhaps noteworthy to mention that the proof of the above theorem remains valid even when an infinite number of generators are adopted. However, in order to execute the proof in its present form, one needs to have the crucial finite intersection property of compact sets.

It is customary to ask about expanding the representations of M-ideals in $H^\infty(\mathbb{D})$ beyond those that are finitely generated or generated by functions from $Z^\infty(\mathbb{D})$. However, as mentioned in Section 3, the set of all M-ideals in $H^\infty(\mathbb{D})$ is vast, and the complexity seems to be akin to that of the maximal ideal spaces of $H^\infty(\mathbb{D})$. Strengthening these results is crucial for advancing the theory of M-ideals and the theory of bounded analytic functions, as it could potentially lead to the development of new approaches in these fields and their connected ones.

We wrap up this paper with a more specific question. In view of Theorem 5.2, we know that all M-ideals are analytic primes. We also note that the classification of closed prime ideals in $H^\infty(\mathbb{D})$ is unknown (however, see [7, 20, 27, 29, 37, 38]). An intriguing question thus emerges concerning the classification of analytic primes in $H^\infty(\mathbb{D})$ or a more general uniform algebra.

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